FERMATEAN NEUTROSOPHIC n-NORMED LINEAR SPACES: CONVERGENCE, COMPLETENESS, AND FIXED POINT APPLICATIONS

MUKHTAR AHMAD[1)] AND FLORENTIN SMARANDACHE[2,*)]

[1)]Doctoral Program of Mathematics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha no. 10, Bandung, 40132, Indonesia
Email: mukhkhu0786@gmail.com; 30125701@mahasiswa.itb.ac.id; Orcid: 0009-0006-4597-2278
[2)]University of New Mexico, Mathematics, Physics and Natural Sciences Division, Gallup, NM 87301, USA
Email: smarand@unm.edu; Orcid: 0000-0002-5560-5926
*corresponding authors email: smarand@unm.edu

Abstract. This paper introduces and investigates Fermatean neutrosophic n-normed linear spaces, a framework that combines the geometric structure of n-normed spaces with truth-membership, indeterminacy-membership, and falsity-membership degrees governed by Fermatean-type cubic admissibility conditions. Fundamental properties of the proposed space are established using continuous t-norms and t-conorms. We define open balls and formulate convergence, Cauchy sequences, and completeness through the asymptotic behavior of the three neutrosophic functions. Equivalent characterizations of convergence and Cauchy sequences are proved, together with the uniqueness of limits and the result that every convergent sequence is Cauchy. Concrete constructions based on the determinant n-norm on $\mathbb{R}^n$ are provided to demonstrate the consistency and applicability of the framework. A Fermatean neutrosophic contraction is then introduced, and a Banach-type fixed point theorem is obtained in complete Fermatean neutrosophic n-normed linear spaces. The theorem guarantees the existence and uniqueness of fixed points for suitable self-mappings. Applications to affine operator equations and nonlinear operator equations are presented, including an iterative approximation procedure. Several examples illustrate the definitions and show how classical constructions arise naturally within the proposed setting. These results extend fuzzy and intuitionistic fuzzy n-normed analysis while providing a flexible mathematical setting for uncertainty, indeterminacy, nonlinear analysis, and fixed point problems.

1. Introduction

Uncertainty, vagueness, incomplete information, and indeterminacy arise naturally in mathematical modelling, engineering, decision theory, economics, and the physical sciences. Classical set theory represents membership through the two values 0 and 1 and therefore does not adequately describe situations in which an object belongs to a set only partially. Zadeh addressed this limitation by introducing fuzzy sets, in which every element is assigned a membership degree from the interval $[0, 1]$ [1]. This theory provided the foundation for numerous extensions of classical analysis, topology, optimization, decision-making, and fixed point theory.

Atanassov extended fuzzy sets by introducing intuitionistic fuzzy sets, in which every element is characterized by a membership degree and a non-membership degree satisfying

$$\mu + \nu \leq 1$$

[2]. Although this condition allows the representation of hesitation, it may be restrictive when the sum of the proposed membership and non-membership degrees exceeds one. Yager introduced Pythagorean fuzzy information, replacing the linear restriction with the quadratic

2020 *Mathematics Subject Classification.* Primary 46B20, 47H10; Secondary 03E72, 47H09, 54A40.
Key words and phrases. Fermatean neutrosophic set; n-normed linear space; neutrosophic norm; convergence; Cauchy sequence; completeness; contraction mapping; fixed point theorem; nonlinear operator equation.

condition

$$\mu^2 + \nu^2 \leq 1$$

[3, 4]. Senapati and Yager subsequently introduced Fermatean fuzzy sets, whose admissible values satisfy

$$\mu^3 + \nu^3 \leq 1$$

[5, 6, 7]. Because the cubic restriction is weaker than the intuitionistic and Pythagorean restrictions, Fermatean fuzzy sets provide a larger admissible region for representing uncertain information.

Following their introduction, Fermatean fuzzy sets have been developed through new algebraic operations, aggregation operators, linguistic models, distance measures, and decision-making methods. Representative contributions include Fermatean fuzzy linguistic sets [8], Fermatean fuzzy TOPSIS methods [9], WASPAS-based supplier evaluation models [10], Yager aggregation operators [11], Einstein aggregation operators [12], interval-valued Fermatean fuzzy sets [13], and Fermatean fuzzy ELECTRE methods [14]. A systematic review by Büyüközkan, Uztürk, and Ilıcak demonstrates the rapid development of Fermatean fuzzy models and their extensions [15]. Nevertheless, most existing Fermatean fuzzy studies concentrate on aggregation and multi-criteria decision-making, while their use in functional analysis, generalized norm structures, convergence theory, and fixed point theory remains comparatively limited.

In another direction, Gähler introduced 2-normed linear spaces as a generalization of ordinary normed spaces [16]. In a 2-normed space, the norm depends on two vectors and geometrically represents the area of the parallelogram generated by those vectors. Misiak subsequently developed n-inner product spaces [17], and Gunawan and Mashadi provided a systematic treatment of n-normed spaces [18]. Additional structural properties and equivalent constructions of n-norms have been investigated in [19]. These spaces are particularly useful for describing linear dependence and volume-type geometric quantities involving several vectors.

The interaction between fuzzy theory and functional analysis produced several forms of fuzzy metric and fuzzy normed spaces. Important foundational contributions include fuzzy vector spaces [27], fuzzy topological vector spaces [28], fuzzy metric spaces [29], and finite-dimensional fuzzy normed linear spaces [31]. Fixed point theory was subsequently developed in fuzzy metric spaces by Grabiec [30], George and Veeramani [32], Gregori and Sapena [33], and Miheţ [34].

Park introduced intuitionistic fuzzy metric spaces using continuous t-norms and t-conorms [35]. Saadati and Park studied the associated intuitionistic fuzzy topological structure [36] and Ahmad and Smarandache [37], while Alaca, Türkoğlu, and Yıldız established fixed point results in this setting [38]. Coupled fixed point results and contraction principles in intuitionistic fuzzy normed spaces were developed in [61]. More recently, Konwar, Esi, and Debnath introduced additional contraction mappings and fixed point results in complete intuitionistic fuzzy normed linear spaces [39].

The concepts of fuzzy and intuitionistic fuzzy norms have also been combined with n-normed geometry. Narayanan and Vijayabalaji introduced fuzzy n-normed linear spaces [20], whereas Vijayabalaji, Thillaigovindan, and Jun developed intuitionistic fuzzy n-normed linear spaces and studied their convergence and Cauchy sequences [21], and Ahmad and Smarandache [22]. Mursaleen and Lohani investigated intuitionistic fuzzy 2-normed spaces [59], and the completion of the related intuitionistic fuzzy 2-metric structure was considered in [60]. Statistical and summability methods in intuitionistic fuzzy n-normed spaces were studied by Sen and Debnath [23], Debnath [24], Konwar and Debnath [25], and, more recently, Debnath [26].

Neutrosophic theory provides another important approach to uncertainty by associating independent truth-membership, indeterminacy-membership, and falsity-membership degrees with each element [40, 41, 42]. Similarity and entropy measures for neutrosophic sets were studied by Majumdar and Samanta [43], while aggregation methods for simplified neutrosophic sets were developed by Ye [44]. Liu and Wang introduced a normalized weighted Bonferroni mean

for single-valued neutrosophic information [45]. These contributions demonstrate the ability of neutrosophic structures to represent incomplete, inconsistent, and indeterminate information that cannot be expressed adequately through a pair of membership and non-membership functions.

Kirişci and Şimşek introduced neutrosophic metric spaces and investigated their topological properties [46] and Ahmad and Mursaleen [47]. They subsequently defined neutrosophic normed spaces and studied statistical convergence in those spaces [48, 49, 50, 51, 52]. Fixed point results in orthogonal neutrosophic metric spaces were established in [53], while Tribonacci convergence and sequence-space constructions in neutrosophic normed spaces were examined in [54, 55]. Nonlinear operators, continuity, boundedness, and Fréchet differentiation between neutrosophic normed spaces were investigated by Khan and Khan [56]. Further fixed point developments include orthogonal neutrosophic 2-metric spaces [57] and stability results for functional equations in neutrosophic normed spaces [58].

These developments show that fuzzy n-normed spaces, Fermatean fuzzy sets, and neutrosophic normed spaces have each received considerable attention [64]. However, the existing theories do not fully combine the following three features: the volume-based geometry of an n-norm, the cubic flexibility of a Fermatean-type structure, and an explicit independent indeterminacy function. In particular, ordinary Fermatean fuzzy models generally use only membership and non-membership functions, while existing neutrosophic normed models do not impose a Fermatean-type cubic structure on their components. This observation motivates the development of Fermatean neutrosophic n-normed linear spaces.

In this paper, we introduce a Fermatean neutrosophic n-normed linear space through three functions

$$\mu_F, \eta_F, \nu_F : X^n \times (0, \infty) \longrightarrow [0, 1],$$

representing truth-membership, indeterminacy-membership, and falsity-membership, respectively. The proposed framework combines the conditions

$$0 \leq \mu_F^3 + \nu_F^3 \leq 1$$

and

$$0 \leq \mu_F^3 + \eta_F^3 + \nu_F^3 \leq 2$$

with the homogeneity, permutation invariance, and triangle-type properties of an n-norm.

The principal contributions of the paper are summarized as follows:

(1) We formulate the axioms of a Fermatean neutrosophic n-normed linear space and provide determinant-based examples on $\mathbb{R}^n$.
(2) We define open balls, convergence, Cauchy sequences, and completeness using the three Fermatean neutrosophic functions.
(3) We establish the uniqueness of limits, equivalent characterizations of convergence and Cauchy sequences, and the fact that every convergent sequence is Cauchy.
(4) We introduce Fermatean neutrosophic contraction mappings and prove a Banach-type fixed point theorem in complete Fermatean neutrosophic n-normed linear spaces.
(5) We apply the fixed point theorem to affine and nonlinear operator equations and illustrate the convergence of the corresponding iterative procedures.

Thus, the proposed theory extends fuzzy and intuitionistic fuzzy n-normed analysis by incorporating an independent indeterminacy degree together with Fermatean-type cubic admissibility. It also provides a foundation for further investigations of operator equations, approximation methods, stability problems, and nonlinear analysis under indeterminate information.

2. Literature Review

2.1. **Fuzzy, intuitionistic fuzzy, and Fermatean fuzzy developments.** The transition from classical sets to fuzzy sets initiated a systematic mathematical treatment of partial membership [1]. Atanassov's intuitionistic fuzzy sets enriched this framework by incorporating non-membership and hesitation [2]. The Pythagorean models of Yager enlarged the admissible domain through a quadratic condition [3, 4], while Fermatean fuzzy sets replaced that condition with a cubic one [7]. The fundamental operations and aggregation mechanisms associated with Fermatean information were studied in [5, 6].

The subsequent Fermatean literature has largely developed in the direction of decision analysis. Linguistic information and aggregation were considered in [8], Dombi-based TOPSIS operators were introduced in [9], and supplier-selection models were studied in [10]. Other important contributions include Yager aggregation operators [11], Einstein aggregation and MULTIMOORA [12], interval-valued Fermatean fuzzy information [13], and ELECTRE-based group decision-making [14]. The systematic review in [15] confirms that aggregation, multi-criteria decision-making, and practical selection problems constitute the dominant themes of Fermatean fuzzy research.

2.2. **Development of fuzzy and intuitionistic fuzzy normed spaces.** The analytical development of fuzzy structures began with fuzzy vector and topological vector spaces [27, 28]. Kaleva and Seikkala developed an influential fuzzy metric framework [29], and Felbin investigated finite-dimensional fuzzy normed linear spaces [31]. The fuzzy Banach contraction principle and related fixed point results were developed through the contributions in [30, 32, 33, 34].

The introduction of intuitionistic fuzzy metric spaces by Park added an explicit non-membership function and a continuous t-conorm to the metric structure [35]. Topological properties were studied by Saadati and Park [36], while fixed point results were developed in [38, 61, 39]. These studies demonstrate that completeness and suitable fuzzy contraction conditions are sufficient to recover analogues of the classical Banach fixed point theorem.

2.3. **Fuzzy and intuitionistic fuzzy n-normed spaces.** The geometric basis of the present work comes from Gähler's 2-normed spaces [16], Misiak's n-inner product spaces [17], and the general theory of n-normed spaces developed by Gunawan and Mashadi [18]. Equivalent n-norm constructions and their relation to ordinary norms were investigated in [19].

Narayanan and Vijayabalaji transferred this geometry to fuzzy analysis by introducing fuzzy n-normed linear spaces [20]. Vijayabalaji, Thillaigovindan, and Jun subsequently incorporated intuitionistic fuzzy membership and non-membership functions [21]. Related 2-normed and 2-metric developments appear in [59, 60]. Convergence and summability in these spaces were examined through statistical convergence [23], lacunary ideal convergence [24], I_λ-convergence [25], and statistical Cesàro summability [26]. These studies provide an important foundation for defining convergence and completeness in generalized fuzzy n-normed spaces.

2.4. **Neutrosophic normed spaces and fixed point theory.** Neutrosophic structures differ from intuitionistic fuzzy structures because truth, indeterminacy, and falsity are treated as three explicit components. Similarity, entropy, aggregation, and decision-making properties of neutrosophic information have been studied in [43, 44, 45]. Kirişci and Şimşek transferred the three-component structure to metric and normed spaces [46, 48].

Recent work has extended neutrosophic analysis in several directions. Orthogonal neutrosophic contraction results were obtained in [53]; neutrosophic sequence spaces and Tribonacci convergence were investigated in [54]; and nonlinear operators and Fréchet differentiation were studied in [56]. Orthogonal neutrosophic 2-metric spaces were developed in [57], while direct and fixed point methods were applied to the stability of functional equations in [58].

2.5. **Research gap.** The reviewed literature reveals a clear separation among Fermatean fuzzy sets, neutrosophic normed spaces, and fuzzy n-normed linear spaces. Fermatean studies provide a flexible cubic admissibility condition but generally lack an independent indeterminacy function. Neutrosophic normed spaces provide the required three-function representation but do not incorporate the Fermatean cubic geometry considered here. Finally, fuzzy and intuitionistic fuzzy n-normed spaces combine uncertainty with volume-type geometry but use only one or two uncertainty functions.

The present work connects these research directions by introducing a three-function Fermatean-type n-normed structure. This formulation makes it possible to develop convergence, completeness, contraction mappings, and fixed point theory while simultaneously retaining truth, indeterminacy, falsity, and the geometric information described by an n-norm.

3. Preliminary Concepts

This section recalls the fundamental concepts required for the development of Fermatean neutrosophic fuzzy n-normed linear spaces. We first review continuous t-norms and t-conorms, followed by the relevant fuzzy and neutrosophic set structures. We then recall the basic notions of n-normed linear spaces, including convergence, Cauchy sequences, and completeness. Finally, we present the intuitionistic fuzzy and neutrosophic n-normed frameworks that motivate the structure introduced in the subsequent section.

3.1. Continuous t-Norms and t-Conorms.

Definition 3.1. *A binary operation*

$$* : [0,1] \times [0,1] \longrightarrow [0,1]$$

is called a continuous t-norm if, for all $a, b, c, d \in [0,1]$, the following conditions hold:

(1) $a * 1 = a$;
(2) $a * b = b * a$;
(3) $(a * b) * c = a * (b * c)$;
(4) *if $a \leq c$ and $b \leq d$, then*
$$a * b \leq c * d;$$
(5) $*$ *is continuous on* $[0,1] \times [0,1]$.

Example 3.1. *Standard examples of continuous t-norms include:*

(1) *the product t-norm,*
$$a * b = ab;$$
(2) *the minimum t-norm,*
$$a * b = \min\{a, b\};$$
(3) *the Lukasiewicz t-norm,*
$$a * b = \max\{a + b - 1, 0\}.$$

Definition 3.2. *A binary operation*

$$\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$$

is called a continuous t-conorm if, for all $a, b, c, d \in [0,1]$, the following conditions hold:

(1) $a \diamond 0 = a$;
(2) $a \diamond b = b \diamond a$;
(3) $(a \diamond b) \diamond c = a \diamond (b \diamond c)$;
(4) *if $a \leq c$ and $b \leq d$, then*
$$a \diamond b \leq c \diamond d;$$
(5) $\diamond$ *is continuous on* $[0,1] \times [0,1]$.

Example 3.2. *Standard examples of continuous t-conorms include:*

(1) *the maximum t-conorm,*

$$a \diamond b = \max\{a, b\};$$

(2) *the probabilistic sum,*

$$a \diamond b = a + b - ab;$$

(3) *the Lukasiewicz t-conorm,*

$$a \diamond b = \min\{a + b, 1\}.$$

Remark 3.1. *Operations compatible with cubic Fermatean degrees can be obtained by applying an ordinary t-conorm to the third powers and then taking a cube root. Examples are*

$$a \diamond b = \left[\min\{a^3 + b^3, 1\}\right]^{1/3}$$

and

$$a \diamond b = \left(a^3 + b^3 - a^3 b^3\right)^{1/3}.$$

3.2. **Fuzzy, Fermatean, and Neutrosophic Set Structures.**

Definition 3.3. *Let X be a nonempty set. A fuzzy set A on X is characterized by a membership function*

$$\mu_A : X \longrightarrow [0, 1],$$

where $\mu_A(x)$ represents the degree to which x belongs to A.

Definition 3.4. *Let X be a nonempty set. An intuitionistic fuzzy set A on X is given by*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A, \nu_A : X \longrightarrow [0, 1]$$

are the membership and non-membership functions, respectively, and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

for every $x \in X$.

Remark 3.2. *The hesitation degree associated with an intuitionistic fuzzy set is*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Thus, $0 \leq \pi_A(x) \leq 1$.

Definition 3.5. *Let X be a nonempty set. A Fermatean fuzzy set A on X is given by*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A, \nu_A : X \longrightarrow [0, 1]$$

satisfy

$$0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1$$

for every $x \in X$.

Remark 3.3. *The Fermatean hesitation degree is defined by*

$$\pi_A(x) = \left(1 - \mu_A^3(x) - \nu_A^3(x)\right)^{1/3}.$$

The Fermatean restriction guarantees that $0 \leq \pi_A(x) \leq 1$.

Definition 3.6. *Let X be a nonempty set. A single-valued neutrosophic set A on X is given by*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A, \eta_A, \nu_A : X \longrightarrow [0, 1]$$

are the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. These functions satisfy

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3$$

for every $x \in X$.

Remark 3.4. *In a neutrosophic set, $\eta_A(x)$ is specified explicitly as the indeterminacy degree of x. In particular, it is not required to be the residual quantity $1 - \mu_A(x) - \nu_A(x)$.*

Definition 3.7. *Let X be a nonempty set. A Fermatean neutrosophic set A on X is given by*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A, \eta_A, \nu_A : X \longrightarrow [0, 1]$$

denote the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, satisfying

$$0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1$$

and

$$0 \leq \eta_A^3(x) \leq 1$$

for every $x \in X$. Consequently,

$$0 \leq \mu_A^3(x) + \eta_A^3(x) + \nu_A^3(x) \leq 2.$$

Remark 3.5. *In Definition 3.7, the truth- and falsity-membership degrees form a Fermatean orthopair, whereas the indeterminacy-membership degree is an additional component. Thus, $\eta_A(x)$ should not be identified with the Fermatean hesitation degree $\pi_A(x)$ unless this equality is explicitly imposed in a particular model.*

3.3. n-Normed Linear Spaces.

Definition 3.8. *Let X be a real linear space with $\dim X \geq n$. A function*

$$\|\cdot, \ldots, \cdot\| : X^n \longrightarrow [0, \infty)$$

is called an n-norm on X if, for all $v_1, \ldots, v_n, w \in X$ and $\alpha \in \mathbb{R}$, the following conditions hold:

(1)
$$\|v_1, \ldots, v_n\| = 0$$
if and only if $v_1, \ldots, v_n$ are linearly dependent;

(2) *$\|v_1, \ldots, v_n\|$ is invariant under every permutation of $v_1, \ldots, v_n$;*

(3)
$$\|v_1, \ldots, v_{n-1}, \alpha v_n\| = |\alpha| \|v_1, \ldots, v_n\|;$$

(4)
$$\|v_1, \ldots, v_{n-1}, v_n + w\| \leq \|v_1, \ldots, v_{n-1}, v_n\| + \|v_1, \ldots, v_{n-1}, w\|.$$

The pair $(X, \|\cdot, \ldots, \cdot\|)$ is then called an n-normed linear space.

Example 3.3. *Let $X = \mathbb{R}^n$. For $v_1, \ldots, v_n \in \mathbb{R}^n$, define*

$$\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|.$$

Then $(\mathbb{R}^n, \|\cdot, \ldots, \cdot\|)$ is an n-normed linear space. Geometrically, this n-norm is the volume of the parallelotope generated by $v_1, \ldots, v_n$.

Definition 3.9. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space. A sequence $\langle x_k \rangle$ in X is said to converge to $x \in X$ if, for every fixed $z_1, \ldots, z_{n-1} \in X$,*

$$\|z_1, \ldots, z_{n-1}, x_k - x\| \longrightarrow 0$$

as $k \to \infty$. In this case, we write $x_k \to x$.

Definition 3.10. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space. A sequence $\langle x_k \rangle$ in X is called a Cauchy sequence if, for every fixed $z_1, \ldots, z_{n-1} \in X$,*

$$\|z_1, \ldots, z_{n-1}, x_k - x_p\| \longrightarrow 0$$

as $k, p \to \infty$.

Definition 3.11. *An n-normed linear space is called complete, or an n-Banach space, if every Cauchy sequence converges to an element of the space with respect to its n-norm.*

3.4. Intuitionistic Fuzzy and Neutrosophic n-Norms.

Definition 3.12. *Let X be a linear space over $\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $*$ be a continuous t-norm and $\diamond$ a continuous t-conorm. A pair of functions*

$$\mu, \nu : X^n \times \mathbb{R} \longrightarrow [0, 1]$$

is called an intuitionistic fuzzy n-norm on X if, for all $v_1, \ldots, v_n, w \in X$, $\alpha \in \mathbb{F}$, $\lambda \in \mathbb{R}$, and $\lambda_1, \lambda_2 > 0$, the following conditions hold:

(1)
$$\mu(v_1, \ldots, v_n, \lambda)\nu(v_1, \ldots, v_n, \lambda) \leq 1;$$

(2) *for $\lambda \leq 0$,*
$$\mu(v_1, \ldots, v_n, \lambda) = 0, \qquad \nu(v_1, \ldots, v_n, \lambda) = 1;$$

(3) *for $\lambda > 0$,*
$$\mu(v_1, \ldots, v_n, \lambda) = 1$$
if and only if $v_1, \ldots, v_n$ are linearly dependent;

(4) *for $\lambda > 0$,*
$$\nu(v_1, \ldots, v_n, \lambda) = 0$$
if and only if $v_1, \ldots, v_n$ are linearly dependent;

(5) *μ and ν are invariant under every permutation of $v_1, \ldots, v_n$;*

(6) *if $\alpha \neq 0$, then*
$$\mu(v_1, \ldots, \alpha v_n, \lambda) = \mu\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right)$$
and
$$\nu(v_1, \ldots, \alpha v_n, \lambda) = \nu\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right);$$

(7)
$$\begin{aligned}\mu(v_1, \ldots, v_{n-1}, v_n + w, \lambda_1 + \lambda_2) \\ \geq \mu(v_1, \ldots, v_n, \lambda_1) * \mu(v_1, \ldots, v_{n-1}, w, \lambda_2);\end{aligned}$$

(8)
$$\begin{aligned}\nu(v_1, \ldots, v_{n-1}, v_n + w, \lambda_1 + \lambda_2) \\ \leq \nu(v_1, \ldots, v_n, \lambda_1) \diamond \nu(v_1, \ldots, v_{n-1}, w, \lambda_2);\end{aligned}$$

(9) $\mu(v_1, \ldots, v_n, \lambda)$ *is non-decreasing in* λ, *and*

$$\lim_{\lambda \to \infty} \mu(v_1, \ldots, v_n, \lambda) = 1;$$

(10) $\nu(v_1, \ldots, v_n, \lambda)$ *is non-increasing in* λ, *and*

$$\lim_{\lambda \to \infty} \nu(v_1, \ldots, v_n, \lambda) = 0.$$

The tuple $(X, \mu, \nu, *, \diamond)$ *is then called an intuitionistic fuzzy n-normed linear space.*

Definition 3.13. *Let* X *be a linear space over* $\mathbb{F}$, *where* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $*$ *be a continuous t-norm and* $\diamond$ *a continuous t-conorm. A triple of functions*

$$\mu, \eta, \nu : X^n \times \mathbb{R} \longrightarrow [0, 1]$$

is called a neutrosophic n-norm on X *if, for all* $v_1, \ldots, v_n, w \in X$, $\alpha \in \mathbb{F}$, $\lambda \in \mathbb{R}$, *and* $\lambda_1, \lambda_2 > 0$, *the following conditions hold:*

(1)
$$0 \leq \mu(v_1, \ldots, v_n, \lambda)\eta(v_1, \ldots, v_n, \lambda) + \nu(v_1, \ldots, v_n, \lambda) \leq 3;$$

(2) *for* $\lambda \leq 0$,
$$\mu(v_1, \ldots, v_n, \lambda) = 0, \qquad \eta(v_1, \ldots, v_n, \lambda) = 1, \qquad \nu(v_1, \ldots, v_n, \lambda) = 1;$$

(3) *for* $\lambda > 0$,
$$\mu(v_1, \ldots, v_n, \lambda) = 1$$
if and only if $v_1, \ldots, v_n$ *are linearly dependent;*

(4) *for* $\lambda > 0$,
$$\eta(v_1, \ldots, v_n, \lambda) = \nu(v_1, \ldots, v_n, \lambda) = 0$$
if and only if $v_1, \ldots, v_n$ *are linearly dependent;*

(5) μ, η, *and* ν *are invariant under every permutation of* $v_1, \ldots, v_n$;

(6) *if* $\alpha \neq 0$, *then*
$$\mu(v_1, \ldots, \alpha v_n, \lambda) = \mu\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right),$$
$$\eta(v_1, \ldots, \alpha v_n, \lambda) = \eta\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right),$$
and
$$\nu(v_1, \ldots, \alpha v_n, \lambda) = \nu\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right);$$

(7)
$$\mu(v_1, \ldots, v_{n-1}, v_n + w, \lambda_1 + \lambda_2) \geq \mu(v_1, \ldots, v_n, \lambda_1) * \mu(v_1, \ldots, v_{n-1}, w, \lambda_2);$$

(8)
$$\eta(v_1, \ldots, v_{n-1}, v_n + w, \lambda_1 + \lambda_2) \leq \eta(v_1, \ldots, v_n, \lambda_1) \diamond \eta(v_1, \ldots, v_{n-1}, w, \lambda_2);$$

(9)
$$\nu(v_1, \ldots, v_{n-1}, v_n + w, \lambda_1 + \lambda_2) \leq \nu(v_1, \ldots, v_n, \lambda_1) \diamond \nu(v_1, \ldots, v_{n-1}, w, \lambda_2);$$

(10) $\mu(v_1, \ldots, v_n, \lambda)$ *is non-decreasing in* λ, *whereas* $\eta(v_1, \ldots, v_n, \lambda)$ *and* $\nu(v_1, \ldots, v_n, \lambda)$ *are non-increasing in* λ;

(11)
$$\lim_{\lambda \to \infty} \mu(v_1, \ldots, v_n, \lambda) = 1,$$
$$\lim_{\lambda \to \infty} \eta(v_1, \ldots, v_n, \lambda) = 0, \qquad \lim_{\lambda \to \infty} \nu(v_1, \ldots, v_n, \lambda) = 0.$$

The tuple $(X, \mu, \eta, \nu, *, \diamond)$ is then called a neutrosophic n-normed linear space.

Example 3.4. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space and consider*

$$a * b = \min\{a, b\}, \qquad a \diamond b = \max\{a, b\}.$$

For $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ and $\lambda > 0$, set

$$N(\widetilde{V}) = \|v_1, \ldots, v_n\|$$

and define

$$\mu(\widetilde{V}, \lambda) = \frac{\lambda}{\lambda + N(\widetilde{V})},$$

$$\eta(\widetilde{V}, \lambda) = \frac{N(\widetilde{V})}{\lambda + N(\widetilde{V})},$$

and

$$\nu(\widetilde{V}, \lambda) = \frac{N(\widetilde{V})}{2\lambda + N(\widetilde{V})}.$$

For $\lambda \leq 0$, set

$$\mu(\widetilde{V}, \lambda) = 0, \qquad \eta(\widetilde{V}, \lambda) = \nu(\widetilde{V}, \lambda) = 1.$$

For $\lambda > 0$,

$$\mu(\widetilde{V}, \lambda) + \eta(\widetilde{V}, \lambda) + \nu(\widetilde{V}, \lambda)$$

$$= 1 + \frac{N(\widetilde{V})}{2\lambda + N(\widetilde{V})} \leq 2.$$

The properties of the underlying n-norm imply that all the remaining conditions of Definition 3.13 are satisfied. Hence, $(X, \mu, \eta, \nu, *, \diamond)$ is a neutrosophic n-normed linear space.

Definition 3.14. *Let X be a nonempty set and let $\tau : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of τ if*

$$\tau(x) = x.$$

The preceding concepts provide the foundation for the Fermatean neutrosophic fuzzy n-normed linear space developed in the next section. In particular, the new structure simultaneously retains the cubic Fermatean restriction on the truth–falsity pair, the explicit indeterminacy component of neutrosophic sets, and the geometric dependence on n vectors inherited from an n-norm.

4. Fermatean Neutrosophic Fuzzy n-Normed Linear Space (FNF-n-NLS)

The theory of fuzzy sets was initiated by Zadeh [1], who introduced a mathematical framework for representing uncertainty through degrees of membership. Later, Atanassov [2] extended this theory by introducing intuitionistic fuzzy sets, in which every element is characterized by both a membership degree and a non-membership degree. This extension provides a more flexible representation of uncertainty, particularly when the available information contains hesitation or incomplete knowledge.

A further generalization was introduced by Smarandache [**?**] through the theory of neutrosophic sets. In a neutrosophic set, each element is described by three components: a truth-membership degree, an indeterminacy-membership degree, and a falsity-membership degree. A fundamental feature of this framework is that the indeterminacy component is represented explicitly rather than being derived only from the truth- and falsity-membership degrees. Consequently, neutrosophic sets provide a broader framework for representing uncertain, incomplete, indeterminate, and potentially inconsistent information.

In functional analysis, the concept of an n-normed linear space provides a natural generalization of ordinary normed and 2-normed linear spaces. The theory of 2-normed linear spaces was developed by Gähler [16], while the more general theory of n-normed linear spaces was subsequently studied by Misiak [17] and by Gunawan and Mashadi [18]. In an n-normed linear space, the size of an object is determined by a function of n vectors. This structure naturally describes linear dependence and geometric volume-like quantities generated by collections of vectors.

The interaction between fuzzy structures and normed linear spaces has led to several important generalizations. Saadati and Park [36] studied intuitionistic fuzzy normed spaces, while Vijayabalaji, Thillaigovindan, and Jun [21] introduced intuitionistic fuzzy n-normed linear spaces and investigated convergence, Cauchy sequences, completeness, and related properties in this setting.

More recently, Senapati and Yager [7] introduced Fermatean fuzzy sets as a broader orthopair fuzzy model. If μ and ν denote the membership and non-membership degrees, respectively, then they satisfy the cubic restriction

$$\mu^3 + \nu^3 \leq 1.$$

This condition is less restrictive than the intuitionistic fuzzy condition

$$\mu + \nu \leq 1.$$

Indeed, every pair satisfying $\mu + \nu \leq 1$ also satisfies $\mu^3 + \nu^3 \leq 1$, but the converse does not necessarily hold. Therefore, Fermatean fuzzy sets allow a wider range of admissible membership and non-membership degrees [64].

The Fermatean neutrosophic framework combines the cubic structure of Fermatean fuzzy sets with the truth, indeterminacy, and falsity components of neutrosophic sets [63]. If μ_F, η_F, and ν_F denote the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively, then the Fermatean neutrosophic restrictions are given by

$$0 \leq \mu_F^3 + \nu_F^3 \leq 1$$

and

$$0 \leq \eta_F^3 \leq 1.$$

Consequently,

$$0 \leq \mu_F^3 + \eta_F^3 + \nu_F^3 \leq 2.$$

Here, the truth- and falsity-membership degrees form a Fermatean orthopair, whereas the indeterminacy-membership degree is represented as a separate component. This additional component enables the model to describe uncertainty and indeterminacy more explicitly than a Fermatean intuitionistic fuzzy structure.

Motivated by these developments, we introduce the notion of a Fermatean neutrosophic fuzzy n-normed linear space (FNF-n-NLS). The proposed structure combines the geometric properties of n-normed linear spaces with the three-component representation of truth, indeterminacy, and falsity under the Fermatean cubic restrictions. It is characterized by three functions

$$\mu_F, \eta_F, \nu_F : X^n \times (0, \infty) \longrightarrow [0, 1],$$

together with a continuous t-norm and a continuous t-conorm.

In this section, we present the formal definition of a Fermatean neutrosophic fuzzy n-normed linear space and investigate its fundamental analytical and topological properties. In particular, we introduce the corresponding open balls, convergence, Cauchy sequences, completeness, contraction mappings, and fixed-point results within the proposed Fermatean neutrosophic framework.

Definition 4.1. *Let X be a linear space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $*$ be a continuous t-norm and let $\diamond$ be a continuous t-conorm. Let*

$$\widetilde{V} = (v_1, \ldots, v_n) \in X^n,$$

and let

$$\mu_N, \eta_N, \nu_N : X^n \times (0, \infty) \to [0, 1]$$

be three functions representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively.

The tuple $\left(X, \mu_N, \eta_N, \nu_N, *, \diamond\right)$ is called a neutrosophic n-normed linear space, briefly an N-n-NLS, if the following conditions are satisfied for all $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$, $\lambda, \lambda_1, \lambda_2 > 0$, $\alpha \in \mathbb{F}$, and $\widetilde{v}_n \in X$:

(1)
$$0 < \mu_N(\widetilde{V}, \lambda) \leq 1, \qquad 0 \leq \eta_N(\widetilde{V}, \lambda) < 1, \qquad 0 \leq \nu_N(\widetilde{V}, \lambda) < 1.$$

(2)
$$0 \leq \mu_N(\widetilde{V}, \lambda) + \eta_N(\widetilde{V}, \lambda) + \nu_N(\widetilde{V}, \lambda) \leq 3.$$

(3)
$$\mu_N(\widetilde{V}, \lambda) = 1$$
if and only if $v_1, \ldots, v_n$ are linearly dependent.

(4)
$$\eta_N(\widetilde{V}, \lambda) = 0$$
if and only if $v_1, \ldots, v_n$ are linearly dependent.

(5)
$$\nu_N(\widetilde{V}, \lambda) = 0$$
if and only if $v_1, \ldots, v_n$ are linearly dependent.

(6) *$\mu_N(\widetilde{V}, \lambda)$ is invariant under any permutation of $v_1, \ldots, v_n$.*

(7) *$\eta_N(\widetilde{V}, \lambda)$ is invariant under any permutation of $v_1, \ldots, v_n$.*

(8) *$\nu_N(\widetilde{V}, \lambda)$ is invariant under any permutation of $v_1, \ldots, v_n$.*

(9) *If $\alpha \neq 0$, then*
$$\mu_N(v_1, \ldots, \alpha v_n, \lambda) = \mu_N\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right).$$

(10) *If $\alpha \neq 0$, then*
$$\eta_N(v_1, \ldots, \alpha v_n, \lambda) = \eta_N\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right).$$

(11) *If $\alpha \neq 0$, then*
$$\nu_N(v_1, \ldots, \alpha v_n, \lambda) = \nu_N\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right).$$

(12)
$$\begin{aligned}\mu_N(v_1, \ldots, v_{n-1}, v_n, \lambda_1) * \mu_N(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \\ \leq \mu_N(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).\end{aligned}$$

(13)
$$\begin{aligned}\eta_N(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \eta_N(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \\ \geq \eta_N(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).\end{aligned}$$

(14)
$$\begin{aligned}\nu_N(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu_N(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \\ \geq \nu_N(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).\end{aligned}$$

(15) *For every fixed* $\widetilde{V} \in X^n$,

$$\mu_N(\widetilde{V}, \cdot) : (0, \infty) \to [0, 1]$$

is continuous.

(16) *For every fixed* $\widetilde{V} \in X^n$,

$$\eta_N(\widetilde{V}, \cdot) : (0, \infty) \to [0, 1]$$

is continuous.

(17) *For every fixed* $\widetilde{V} \in X^n$,

$$\nu_N(\widetilde{V}, \cdot) : (0, \infty) \to [0, 1]$$

is continuous.

Example 4.1. *Let* $X = \mathbb{R}^n$ *and define the usual* n*-norm on* X *by* $\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|$, *where* $v_1, \ldots, v_n \in \mathbb{R}^n$.

Let

$$a * b = ab$$

be the product t*-norm, and let*

$$a \diamond b = a + b - ab$$

be the probabilistic sum t*-conorm, for all* $a, b \in [0, 1]$.

For $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ *and* $\lambda > 0$, *define*

$$\mu_F(\widetilde{V}, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|} \right)^{\frac{1}{3}},$$

$$\eta_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|} \right)^{\frac{1}{3}},$$

and

$$\nu_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|} \right)^{\frac{1}{3}}.$$

Then, $\left(X^n, \mu_F, \eta_F, \nu_F, *, \diamond\right)$ *is a Fermatean neutrosophic fuzzy* n*-normed linear space (FNF-n-NLS).*

Proof. Let

$$N(\widetilde{V}) = \|v_1, \ldots, v_n\|.$$

Since $N(\widetilde{V}) \geq 0$, for every $\lambda > 0$ we have

$$0 < \frac{\lambda}{\lambda + N(\widetilde{V})} \leq 1,$$

$$0 \leq \frac{N(\widetilde{V})}{2\lambda + N(\widetilde{V})} < 1,$$

and

$$0 \leq \frac{N(\widetilde{V})}{\lambda + N(\widetilde{V})} < 1.$$

Therefore,

$$0 < \mu_F(\widetilde{V}, \lambda) \leq 1,$$

$$0 \leq \eta_F(\widetilde{V}, \lambda) < 1,$$

and

$$0 \leq \nu_F(\widetilde{V}, \lambda) < 1.$$

Moreover,

$$\mu_F^3(\widetilde{V},\lambda)=\frac{\lambda}{\lambda+N(\widetilde{V})}$$

and

$$\nu_F^3(\widetilde{V},\lambda)=\frac{N(\widetilde{V})}{\lambda+N(\widetilde{V})}.$$

Hence,

$$\mu_F^3(\widetilde{V},\lambda)+\nu_F^3(\widetilde{V},\lambda)=1.$$

Also,

$$0\leq\eta_F^3(\widetilde{V},\lambda)=\frac{N(\widetilde{V})}{2\lambda+N(\widetilde{V})}<1.$$

Consequently,

$$\begin{aligned}&\mu_F^3(\widetilde{V},\lambda)+\eta_F^3(\widetilde{V},\lambda)+\nu_F^3(\widetilde{V},\lambda)\\&=1+\frac{N(\widetilde{V})}{2\lambda+N(\widetilde{V})}\leq 2.\end{aligned}$$

Thus, the Fermatean neutrosophic conditions are satisfied.

Next,

$$\mu_F(\widetilde{V},\lambda)=1$$

if and only if

$$\frac{\lambda}{\lambda+N(\widetilde{V})}=1,$$

which is equivalent to

$$N(\widetilde{V})=0.$$

Similarly,

$$\eta_F(\widetilde{V},\lambda)=0\quad\Longleftrightarrow\quad N(\widetilde{V})=0$$

and

$$\nu_F(\widetilde{V},\lambda)=0\quad\Longleftrightarrow\quad N(\widetilde{V})=0.$$

Since

$$N(\widetilde{V})=\|v_1,\ldots,v_n\|=0$$

if and only if $v_1,\ldots,v_n$ are linearly dependent, we obtain

$$\mu_F(\widetilde{V},\lambda)=1$$

if and only if $v_1,\ldots,v_n$ are linearly dependent. The same linear-dependence condition is characterized by $\eta_F(\widetilde{V},\lambda)=0$ and $\nu_F(\widetilde{V},\lambda)=0$.

Since the ordinary n-norm is invariant under every permutation of its variables, the functions μ_F, η_F, and ν_F are also invariant under every permutation of $v_1,\ldots,v_n$.

Let $\alpha\neq 0$. Since

$$\|v_1,\ldots,\alpha v_n\|=|\alpha|\|v_1,\ldots,v_n\|,$$

we obtain

$$\begin{aligned}\mu_F(v_1,\ldots,\alpha v_n,\lambda)&=\left(\frac{\lambda}{\lambda+|\alpha|N(\widetilde{V})}\right)^{\frac{1}{3}}\\&=\mu_F\left(v_1,\ldots,v_n,\frac{\lambda}{|\alpha|}\right).\end{aligned}$$

Furthermore,

$$\eta_F(v_1,\dots,\alpha v_n,\lambda) = \left(\frac{|\alpha|N(\widetilde{V})}{2\lambda+|\alpha|N(\widetilde{V})}\right)^{\frac{1}{3}} = \eta_F\left(v_1,\dots,v_n,\frac{\lambda}{|\alpha|}\right),$$

and

$$\nu_F(v_1,\dots,\alpha v_n,\lambda) = \left(\frac{|\alpha|N(\widetilde{V})}{\lambda+|\alpha|N(\widetilde{V})}\right)^{\frac{1}{3}} = \nu_F\left(v_1,\dots,v_n,\frac{\lambda}{|\alpha|}\right).$$

Thus, the homogeneity conditions are satisfied.
It remains to verify the triangle-type conditions. Set

$$A = \|v_1,\dots,v_{n-1},v_n\|,$$
$$B = \|v_1,\dots,v_{n-1},\widetilde{v}_n\|,$$

and

$$C = \|v_1,\dots,v_{n-1},v_n+\widetilde{v}_n\|.$$

By the triangle inequality for the ordinary n-norm,

$$C \le A + B.$$

We first note that, for all $x, y > 0$,

$$\frac{x}{x+A}\frac{y}{y+B} \le \frac{x+y}{x+y+A+B}.$$

Indeed,

$$\frac{x+y}{x+y+A+B} - \frac{x}{x+A}\frac{y}{y+B} = \frac{x^2B+y^2A+(x+y)AB}{(x+y+A+B)(x+A)(y+B)} \ge 0.$$

Taking $x=\lambda_1$ and $y=\lambda_2$, and using $C \le A+B$, we obtain

$$\frac{\lambda_1}{\lambda_1+A}\frac{\lambda_2}{\lambda_2+B} \le \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+C}.$$

Therefore,

$$\mu_F^3(v_1,\dots,v_{n-1},v_n,\lambda_1) * \mu_F^3(v_1,\dots,v_{n-1},\widetilde{v}_n,\lambda_2) \le \mu_F^3(v_1,\dots,v_{n-1},v_n+\widetilde{v}_n,\lambda_1+\lambda_2).$$

For the falsity function, let

$$p_1 = \frac{\lambda_1}{\lambda_1+A}, \qquad p_2 = \frac{\lambda_2}{\lambda_2+B},$$

and

$$p_{12} = \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+C}.$$

It gives

$$p_1p_2 \le p_{12}.$$

Since

$$\nu_F^3(v_1,\dots,v_{n-1},v_n,\lambda_1) = 1-p_1$$

and

$$\nu_F^3(v_1,\dots,v_{n-1},\widetilde{v}_n,\lambda_2) = 1-p_2,$$

we have

$$(1-p_1)\diamond(1-p_2) = 1-p_1p_2 \ge 1-p_{12}.$$

Consequently,

$$\nu_F^3(v_1,\dots,v_{n-1},v_n,\lambda_1)\diamond\nu_F^3(v_1,\dots,v_{n-1},\widetilde{v}_n,\lambda_2)\geq\nu_F^3(v_1,\dots,v_{n-1},v_n+\widetilde{v}_n,\lambda_1+\lambda_2).$$

For the indeterminacy function, take

$$x=2\lambda_1\quad\text{and}\quad y=2\lambda_2.$$

The same argument yields

$$\frac{2\lambda_1}{2\lambda_1+A}\frac{2\lambda_2}{2\lambda_2+B}\leq\frac{2(\lambda_1+\lambda_2)}{2(\lambda_1+\lambda_2)+C}.$$

Define

$$r_1=\frac{2\lambda_1}{2\lambda_1+A},\qquad r_2=\frac{2\lambda_2}{2\lambda_2+B},$$

and

$$r_{12}=\frac{2(\lambda_1+\lambda_2)}{2(\lambda_1+\lambda_2)+C}.$$

Then $r_1r_2\leq r_{12}$. Moreover,

$$\eta_F^3(v_1,\dots,v_{n-1},v_n,\lambda_1)=1-r_1$$

and

$$\eta_F^3(v_1,\dots,v_{n-1},\widetilde{v}_n,\lambda_2)=1-r_2.$$

Thus,

$$(1-r_1)\diamond(1-r_2)=1-r_1r_2\geq1-r_{12}.$$

Hence,

$$\eta_F^3(v_1,\dots,v_{n-1},v_n,\lambda_1)\diamond\eta_F^3(v_1,\dots,v_{n-1},\widetilde{v}_n,\lambda_2)\geq\eta_F^3(v_1,\dots,v_{n-1},v_n+\widetilde{v}_n,\lambda_1+\lambda_2).$$

Finally, μ_F, η_F, and ν_F are compositions of continuous functions of $\lambda>0$. Therefore, all three functions are continuous on $(0,\infty)$. They also satisfy

$$\lim_{\lambda\to\infty}\mu_F(\widetilde{V},\lambda)=1,$$

$$\lim_{\lambda\to\infty}\eta_F(\widetilde{V},\lambda)=0,\qquad\lim_{\lambda\to\infty}\nu_F(\widetilde{V},\lambda)=0.$$

Therefore, $\big(X^n,\mu_F,\eta_F,\nu_F,*,\diamond\big)$ is a Fermatean neutrosophic fuzzy n-normed linear space. □

Definition 4.2. *Let $\mathcal{A}=\big(X,\mu_N,\eta_N,\nu_N,*,\diamond\big)$ be a neutrosophic n-normed linear space, where $\widetilde{V}=(v_1,v_2,\dots,v_n)\in X^n$ and $\lambda\in(0,\infty)$.*

(i) *If $\|\cdot\|_N$ is a norm previously defined on X^n, then the norm-open ball with center $\widetilde{\alpha}\in X^n$ and radius $r>0$ is defined by*

$$B_N(\widetilde{\alpha},r)=\left\{\widetilde{\beta}\in X^n:\|\widetilde{\alpha}-\widetilde{\beta}\|_N<r\right\}.$$

(ii) *For $\widetilde{V}\in X^n$, $0<r<1$, and $\lambda>0$, the neutrosophic open ball with center $\widetilde{V}$, radius r, and parameter λ is defined by*

$$\begin{aligned}B_N(\widetilde{V},r,\lambda)=\big\{\widetilde{W}\in X^n:\ &\mu_N(\widetilde{V}-\widetilde{W},\lambda)>1-r,\\ &\eta_N(\widetilde{V}-\widetilde{W},\lambda)<r,\\ &\nu_N(\widetilde{V}-\widetilde{W},\lambda)<r\big\},\end{aligned}$$

where

$$\widetilde{V}-\widetilde{W}=(v_1-w_1,v_2-w_2,\dots,v_n-w_n).$$

Here,

$$\mu_N(\widetilde{V}-\widetilde{W},\lambda)>1-r$$

means that $\widetilde{W}$ has a sufficiently high truth-membership degree of closeness to $\widetilde{V}$, while

$$\eta_N(\widetilde{V}-\widetilde{W},\lambda)<r$$

means that the indeterminacy degree of closeness is sufficiently small, and

$$\nu_N(\widetilde{V}-\widetilde{W},\lambda)<r$$

means that the falsity-membership degree of closeness is sufficiently small.

Example 4.2. *Let $X=\mathbb{R}^2$ and let $n=2$. For $\widetilde{U}=(u_1,u_2)\in X^2$, define by*

$$N(\widetilde{U})=\|u_1,u_2\|=\left|\det(u_1,u_2)\right|.$$

Consider the neutrosophic functions

$$\mu_N(\widetilde{U},\lambda)=\left(\frac{\lambda}{\lambda+N(\widetilde{U})}\right)^{\frac{1}{3}},$$

$$\eta_N(\widetilde{U},\lambda)=\left(\frac{N(\widetilde{U})}{2\lambda+N(\widetilde{U})}\right)^{\frac{1}{3}},$$

and

$$\nu_N(\widetilde{U},\lambda)=\left(\frac{N(\widetilde{U})}{\lambda+N(\widetilde{U})}\right)^{\frac{1}{3}}.$$

Let

$$a*b=ab\qquad and\qquad a\diamond b=a+b-ab.$$

Then,

$$\mathcal{A}=\big(X,\mu_N,\eta_N,\nu_N,*,\diamond\big)$$

is a neutrosophic 2-normed linear space.

Consider the center

$$\widetilde{V}=(e_1,e_2),\ \text{where } e_1=(1,0)\qquad and\qquad e_2=(0,1).$$

First, define a norm on X^2 by

$$\|\widetilde{U}\|_N=\sqrt{\|u_1\|_2^2+\|u_2\|_2^2},$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^2$. The norm-open ball with center $\widetilde{V}$ and radius $\frac{1}{2}$ is

$$B_N\left(\widetilde{V},\frac{1}{2}\right)=\left\{\widetilde{W}=(w_1,w_2)\in X^2:\sqrt{\|e_1-w_1\|_2^2+\|e_2-w_2\|_2^2}<\frac{1}{2}\right\}.$$

Now, we consider the neutrosophic open ball with

$$r=\frac{1}{2}\qquad and\qquad \lambda=1.$$

For $\widetilde{W}=(w_1,w_2)\in X^2$, set

$$D(\widetilde{W})=\left|\det(e_1-w_1,e_2-w_2)\right|.$$

Then,

$$\widetilde{W}\in B_N\left(\widetilde{V},\frac{1}{2},1\right)$$

if and only if

$$\left(\frac{1}{1+D(\widetilde{W})}\right)^{\frac{1}{3}} > \frac{1}{2},$$

$$\left(\frac{D(\widetilde{W})}{2+D(\widetilde{W})}\right)^{\frac{1}{3}} < \frac{1}{2},$$

and

$$\left(\frac{D(\widetilde{W})}{1+D(\widetilde{W})}\right)^{\frac{1}{3}} < \frac{1}{2}.$$

The first inequality is equivalent to

$$D(\widetilde{W}) < 7.$$

The second inequality is equivalent to

$$D(\widetilde{W}) < \frac{2}{7},$$

while the third inequality is equivalent to

$$D(\widetilde{W}) < \frac{1}{7}.$$

Therefore, all three conditions hold precisely when

$$D(\widetilde{W}) < \frac{1}{7}.$$

Hence,

$$\boxed{B_N\left(\widetilde{V}, \frac{1}{2}, 1\right) = \left\{(w_1, w_2) \in X^2 : |\det(e_1 - w_1, e_2 - w_2)| < \frac{1}{7}\right\}.}$$

For example, let

$$\widetilde{W_1} = \big((1.05, 0), (0, 1.05)\big).$$

Then,

$$e_1 - (1.05, 0) = (-0.05, 0)$$

and

$$e_2 - (0, 1.05) = (0, -0.05).$$

Thus,

$$D(\widetilde{W_1}) = \left|\det\begin{pmatrix} -0.05 & 0 \\ 0 & -0.05 \end{pmatrix}\right| = 0.0025 < \frac{1}{7}.$$

Therefore,

$$\widetilde{W_1} \in B_N\left(\widetilde{V}, \frac{1}{2}, 1\right).$$

On the other hand, let

$$\widetilde{W_2} = \big((1.5, 0), (0, 1.5)\big).$$

Then,

$$D(\widetilde{W_2}) = \left|\det\begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}\right| = \frac{1}{4} > \frac{1}{7}.$$

In particular,

$$\nu_N\left(\widetilde{V} - \widetilde{W_2}, 1\right) = \left(\frac{\frac{1}{4}}{1+\frac{1}{4}}\right)^{\frac{1}{3}} = \left(\frac{1}{5}\right)^{\frac{1}{3}} > \frac{1}{2}.$$

Therefore,

$$\widetilde{W_2} \notin B_N\left(\widetilde{V}, \frac{1}{2}, 1\right).$$

This example shows that membership in a neutrosophic open ball is determined simultaneously by a sufficiently large truth-membership degree and sufficiently small indeterminacy and falsity-membership degrees.

Definition 4.3. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space. A sequence $\langle x_k \rangle$ in X is said to converge to $x \in X$ with respect to $\mathcal{A}$ if*

$$\|x_k - x\|_{FIF} \to 0 \quad as \quad k \to \infty.$$

Equivalently, $\langle x_k \rangle$ converges to x if, for every $\varepsilon \in (0,1)$ and every $\lambda > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > (1-\varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k \geq k_0$ and for all $z_1, \ldots, z_{n-1} \in X$.

In this case, we write

$$x_k \longrightarrow x \quad in\ \mathcal{A},$$

or

$$\lim_{k\to\infty} x_k = x.$$

Since this is a Fermatean structure, the same condition may also be written in cubic form as:

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon.$$

Example 4.3. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by $\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|$, where $v_1, \ldots, v_n \in \mathbb{R}^n$.*

Let

$$a * b = ab$$

be the product t-norm, and let

$$a \diamond b = a + b - ab$$

be the probabilistic sum t-conorm, for all $a, b \in [0,1]$.

For

$$\widetilde{V} = (v_1, \ldots, v_n) \in X^n$$

and $\lambda > 0$, define

$$\mu_F(\widetilde{V}, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

$$\eta_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Then, $\left(X^n, \mu_F, \eta_F, \nu_F, *, \diamond\right)$ is a Fermatean neutrosophic fuzzy n-normed linear space.

Now fix a vector $u \in \mathbb{R}^n$ and define a sequence $\langle x_k \rangle$ in X by

$$x_k = \frac{u}{k}, \qquad k \in \mathbb{N}.$$

We claim that $x_k \longrightarrow 0$ in the Fermatean neutrosophic fuzzy n-normed linear space.

Proof. First, for every

$$\widetilde{V} = (v_1, \ldots, v_n) \in X^n$$

and $\lambda > 0$, we have

$$\mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) = \frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|} + \frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|} = 1.$$

Moreover,

$$0 \leq \eta_F^3(\widetilde{V}, \lambda) = \frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|} < 1.$$

Consequently,

$$\mu_F^3(\widetilde{V}, \lambda) + \eta_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) = 1 + \frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|} < 2.$$

Thus, the Fermatean neutrosophic restrictions are satisfied.

We now prove the convergence of $\langle x_k \rangle$. Let $\varepsilon \in (0, 1)$ and $\lambda > 0$ be given, and let

$$z_1, \ldots, z_{n-1} \in X$$

be fixed. Since

$$x_k = \frac{u}{k},$$

we have

$$\begin{aligned} \|z_1, \ldots, z_{n-1}, x_k - 0\| &= \left\|z_1, \ldots, z_{n-1}, \frac{u}{k}\right\| \\ &= \frac{1}{k}\|z_1, \ldots, z_{n-1}, u\|, \end{aligned}$$

where the last equality follows from the homogeneity of the ordinary n-norm.

Now, Let

$$C = \|z_1, \ldots, z_{n-1}, u\|.$$

Then,

$$\|z_1, \ldots, z_{n-1}, x_k\| = \frac{C}{k}.$$

Therefore,

$$\begin{aligned} \mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) &= \frac{\lambda}{\lambda + \frac{C}{k}} \\ &= \frac{k\lambda}{k\lambda + C}, \\ \eta_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) &= \frac{\frac{C}{k}}{2\lambda + \frac{C}{k}} \\ &= \frac{C}{2k\lambda + C}, \end{aligned}$$

and

$$\begin{aligned} \nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) &= \frac{\frac{C}{k}}{\lambda + \frac{C}{k}} \\ &= \frac{C}{k\lambda + C}. \end{aligned}$$

First, we verify the truth-membership condition. We need

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) > 1 - \varepsilon.$$

This is equivalent to

$$\frac{k\lambda}{k\lambda + C} > 1 - \varepsilon.$$

Therefore,

$$k\lambda > (1-\varepsilon)(k\lambda + C),$$

or equivalently,

$$\varepsilon k\lambda > (1-\varepsilon)C.$$

Thus, the truth-membership condition holds whenever

$$k > \frac{(1-\varepsilon)C}{\varepsilon\lambda}.$$

Next, we verify the indeterminacy condition. We need

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon.$$

This is equivalent to

$$\frac{C}{2k\lambda + C} < \varepsilon.$$

Consequently,

$$C < \varepsilon(2k\lambda + C),$$

which gives

$$(1-\varepsilon)C < 2\varepsilon k\lambda.$$

Thus, the indeterminacy condition holds whenever

$$k > \frac{(1-\varepsilon)C}{2\varepsilon\lambda}.$$

Finally, we verify the falsity-membership condition. We need

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon.$$

This is equivalent to

$$\frac{C}{k\lambda + C} < \varepsilon.$$

Therefore,

$$C < \varepsilon(k\lambda + C),$$

and hence

$$(1-\varepsilon)C < \varepsilon k\lambda.$$

Thus, the falsity-membership condition holds whenever

$$k > \frac{(1-\varepsilon)C}{\varepsilon\lambda}.$$

Since

$$\frac{(1-\varepsilon)C}{2\varepsilon\lambda} \leq \frac{(1-\varepsilon)C}{\varepsilon\lambda},$$

it is sufficient to choose

$$k_0 = \left\lfloor \frac{(1-\varepsilon)C}{\varepsilon\lambda} \right\rfloor + 1.$$

Then, for every $k \geq k_0$, we have

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) > 1-\varepsilon,$$
$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon.$$

Since the cube-root function is strictly increasing on $[0,1]$, the preceding inequalities are equivalent to

$$\mu_F(z_1, \ldots, z_{n-1}, x_k, \lambda) > (1-\varepsilon)^{\frac{1}{3}},$$
$$\eta_F(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon^{\frac{1}{3}}.$$

Since $\varepsilon \in (0,1)$, $\lambda > 0$, and the fixed vectors $z_1, \ldots, z_{n-1} \in X$ were arbitrary, it follows that

$$x_k \longrightarrow 0$$

in the Fermatean neutrosophic fuzzy n-normed linear space. □

Definition 4.4. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space. A sequence $\langle x_k \rangle$ in X is said to converge to $x \in X$ with respect to $\mathcal{A}$ if, for every $\varepsilon \in (0,1)$, every $\lambda > 0$, and every $z_1, \ldots, z_{n-1} \in X$, there exists*

$$k_0 = k_0(\varepsilon, \lambda, z_1, \ldots, z_{n-1}) \in \mathbb{N}$$

such that

$$\mu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > (1-\varepsilon)^{\frac{1}{3}},$$

$$\eta_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon^{\frac{1}{3}}$$

for every $k \geq k_0$.

In this case, we write

$$x_k \longrightarrow x \quad \text{in } \mathcal{A},$$

or

$$\lim_{k\to\infty} x_k = x.$$

Since $\mathcal{A}$ has a Fermatean-type structure, the preceding conditions can equivalently be written as

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon,$$

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon.$$

Equivalently, for every $\lambda > 0$ and every $z_1, \ldots, z_{n-1} \in X$,

$$\lim_{k\to\infty} \mu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = 1,$$

$$\lim_{k\to\infty} \eta_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = 0,$$

and

$$\lim_{k\to\infty} \nu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = 0.$$

Example 4.4. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by $\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|$, where $v_1, \ldots, v_n \in \mathbb{R}^n$. Let*

$$a * b = ab$$

be the product t-norm, and let

$$a \diamond b = a + b - ab$$

be the probabilistic sum t-conorm, for all $a, b \in [0,1]$.

For $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ and $\lambda > 0$, define

$$\mu_F(\widetilde{V}, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

$$\eta_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(\widetilde{V},\lambda)=\left(\frac{\|v_1,\dots,v_n\|}{\lambda+\|v_1,\dots,v_n\|}\right)^{\frac{1}{3}}.$$

Then, $\mathcal{A}=\big(X,\mu_F,\eta_F,\nu_F,*,\diamond\big)$ *is a Fermatean neutrosophic* n*-normed linear space.*

Fix $x,u\in X$*, where* $u\neq 0$*, and define the sequence* $\langle x_k\rangle$ *by*

$$x_k=x+\frac{u}{k^2},\qquad k\in\mathbb{N}.$$

Then, $x_k\longrightarrow x\qquad$ *in* $\mathcal{A}$*.*

Proof. Let $\varepsilon\in(0,1)$ and $\lambda>0$ be given, and let

$$z_1,\dots,z_{n-1}\in X$$

be fixed. Since

$$x_k-x=\frac{u}{k^2},$$

the homogeneity of the ordinary n-norm gives

$$\begin{aligned}\|z_1,\dots,z_{n-1},x_k-x\|&=\left\|z_1,\dots,z_{n-1},\frac{u}{k^2}\right\|\\&=\frac{1}{k^2}\|z_1,\dots,z_{n-1},u\|.\end{aligned}$$

Set

$$C=\|z_1,\dots,z_{n-1},u\|.$$

Then,

$$\|z_1,\dots,z_{n-1},x_k-x\|=\frac{C}{k^2}.$$

Therefore,

$$\begin{aligned}\mu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)&=\frac{\lambda}{\lambda+\frac{C}{k^2}}\\&=\frac{k^2\lambda}{k^2\lambda+C},\\\eta_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)&=\frac{\frac{C}{k^2}}{2\lambda+\frac{C}{k^2}}\\&=\frac{C}{2k^2\lambda+C},\end{aligned}$$

and

$$\begin{aligned}\nu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)&=\frac{\frac{C}{k^2}}{\lambda+\frac{C}{k^2}}\\&=\frac{C}{k^2\lambda+C}.\end{aligned}$$

Choose

$$k_0=\left\lfloor\sqrt{\frac{(1-\varepsilon)C}{\varepsilon\lambda}}\right\rfloor+1.$$

Then, for every $k\geq k_0$,

$$k^2>\frac{(1-\varepsilon)C}{\varepsilon\lambda}.$$

Hence,

$$\varepsilon k^2\lambda>(1-\varepsilon)C.$$

For the truth-membership function, we have

$$\begin{aligned}&\mu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)-(1-\varepsilon)\\&=\frac{k^2\lambda}{k^2\lambda+C}-(1-\varepsilon)\\&=\frac{\varepsilon k^2\lambda-(1-\varepsilon)C}{k^2\lambda+C}>0.\end{aligned}$$

Therefore,

$$\mu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)>1-\varepsilon.$$

For the indeterminacy function, we have

$$\begin{aligned}&\varepsilon-\eta_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)\\&=\varepsilon-\frac{C}{2k^2\lambda+C}\\&=\frac{2\varepsilon k^2\lambda-(1-\varepsilon)C}{2k^2\lambda+C}.\end{aligned}$$

Since

$$\varepsilon k^2\lambda>(1-\varepsilon)C,$$

we also have

$$2\varepsilon k^2\lambda>(1-\varepsilon)C.$$

Thus,

$$\eta_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)<\varepsilon.$$

For the falsity-membership function, we obtain

$$\begin{aligned}&\varepsilon-\nu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)\\&=\varepsilon-\frac{C}{k^2\lambda+C}\\&=\frac{\varepsilon k^2\lambda-(1-\varepsilon)C}{k^2\lambda+C}>0.\end{aligned}$$

Therefore,

$$\nu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)<\varepsilon.$$

Letting cube roots, we obtain

$$\mu_F(z_1,\dots,z_{n-1},x_k-x,\lambda)>(1-\varepsilon)^{\frac{1}{3}},$$

$$\eta_F(z_1,\dots,z_{n-1},x_k-x,\lambda)<\varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1,\dots,z_{n-1},x_k-x,\lambda)<\varepsilon^{\frac{1}{3}}$$

for every $k\geq k_0$.

Equivalently,

$$\lim_{k\to\infty}\mu_F(z_1,\dots,z_{n-1},x_k-x,\lambda)=1,$$

$$\lim_{k\to\infty}\eta_F(z_1,\dots,z_{n-1},x_k-x,\lambda)=0,$$

and

$$\lim_{k\to\infty}\nu_F(z_1,\dots,z_{n-1},x_k-x,\lambda)=0.$$

Since $\varepsilon\in(0,1)$, $\lambda>0$, and the fixed vectors $z_1,\dots,z_{n-1}\in X$ were arbitrary, Definition 4.4 implies that

$$x_k\longrightarrow x\quad\text{in }\mathcal{A}.$$

□

Theorem 4.1. *Every intuitionistic fuzzy normed linear space naturally determines a Fermatean neutrosophic normed linear space.*

Proof. Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed linear space. Define three functions

$$\mu_F, \eta_F, \nu_F : X \times (0, \infty) \to [0, 1]$$

by

$$\mu_F(x,t) = \mu(x,t)^{\frac{1}{3}}, \qquad \eta_F(x,t) = \nu(x,t)^{\frac{1}{3}}, \qquad \nu_F(x,t) = \nu(x,t)^{\frac{1}{3}}.$$

Then,

$$\mu_F^3(x,t) = \mu(x,t), \qquad \eta_F^3(x,t) = \nu(x,t), \qquad \nu_F^3(x,t) = \nu(x,t).$$

Since

$$\mu(x,t) + \nu(x,t) \leq 1,$$

we obtain

$$\mu_F^3(x,t) + \nu_F^3(x,t) = \mu(x,t) + \nu(x,t) \leq 1.$$

Thus, the Fermatean admissibility condition is satisfied. Moreover,

$$\begin{aligned} &\mu_F^3(x,t) + \eta_F^3(x,t) + \nu_F^3(x,t) \\ &\quad = \mu(x,t) + 2\nu(x,t) \\ &\quad = \mu(x,t) + \nu(x,t) + \nu(x,t) \leq 2, \end{aligned}$$

because

$$\mu(x,t) + \nu(x,t) \leq 1 \quad \text{and} \quad 0 \leq \nu(x,t) \leq 1.$$

Hence,

$$0 \leq \mu_F^3(x,t) + \eta_F^3(x,t) + \nu_F^3(x,t) \leq 2.$$

Furthermore,

$$\mu_F(x,t) = 1 \iff \mu(x,t) = 1 \iff x = 0,$$

and

$$\eta_F(x,t) = 0 \iff \nu(x,t) = 0 \iff x = 0.$$

Similarly,

$$\nu_F(x,t) = 0 \iff x = 0.$$

If $\alpha \neq 0$, then

$$\begin{aligned} \mu_F(\alpha x, t) &= \mu(\alpha x, t)^{\frac{1}{3}} \\ &= \mu\left(x, \frac{t}{|\alpha|}\right)^{\frac{1}{3}} \\ &= \mu_F\left(x, \frac{t}{|\alpha|}\right). \end{aligned}$$

In the same way,

$$\eta_F(\alpha x, t) = \eta_F\left(x, \frac{t}{|\alpha|}\right)$$

and

$$\nu_F(\alpha x, t) = \nu_F\left(x, \frac{t}{|\alpha|}\right).$$

For the Fermatean triangle condition, we have

$$\begin{aligned} \mu_F^3(x+y, t+s) &= \mu(x+y, t+s) \\ &\geq \mu(x,t) * \mu(y,s) \\ &= \mu_F^3(x,t) * \mu_F^3(y,s). \end{aligned}$$

Moreover,

$$\begin{aligned}\eta_F^3(x+y,t+s) &= \nu(x+y,t+s)\\ &\le \nu(x,t)\diamond\nu(y,s)\\ &= \eta_F^3(x,t)\diamond\eta_F^3(y,s),\end{aligned}$$

and similarly,

$$\nu_F^3(x+y,t+s) \le \nu_F^3(x,t)\diamond\nu_F^3(y,s).$$

Finally, since the function $u \mapsto u^{1/3}$ is continuous and increasing on $[0,1]$, the required continuity and limiting properties of μ_F, η_F, and ν_F follow directly from those of μ and ν.

Therefore, $(X, \mu_F, \eta_F, \nu_F, *, \diamond)$ is a Fermatean neutrosophic normed linear space. □

Theorem 4.2. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space. If a sequence $\langle x_k\rangle$ in X converges to $x \in X$ and also converges to $y \in X$ in the sense of Definition 4.4, then*

$$x = y.$$

Therefore, the limit of a convergent sequence in a Fermatean neutrosophic n-normed linear space is unique.

Proof. Suppose that $\langle x_k\rangle$ converges to both x and y in the Fermatean neutrosophic n-normed linear space

$$\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big).$$

Thus,

$$x_k \longrightarrow x \qquad \text{and} \qquad x_k \longrightarrow y$$

as $k \to \infty$.

Assume, to the contrary, that

$$x \neq y.$$

Then, $x - y \neq 0$. Since X is an n-normed linear space, there exist $z_1, \ldots, z_{n-1} \in X$ such that

$$z_1, \ldots, z_{n-1}, x - y$$

are linearly independent.

Fix $\lambda > 0$ and let $\varepsilon \in (0,1)$. Since $*$ is a continuous t-norm and

$$1 * 1 = 1,$$

there exists $\delta_1 \in (0,1)$ such that

$$(1-\delta_1) * (1-\delta_1) > 1 - \varepsilon.$$

Similarly, since $\diamond$ is a continuous t-conorm and

$$0 \diamond 0 = 0,$$

there exists $\delta_2 \in (0,1)$ such that

$$\delta_2 \diamond \delta_2 < \varepsilon.$$

Set

$$\delta = \min\{\delta_1, \delta_2\}.$$

By the monotonicity of $*$ and $\diamond$, we have

$$(1-\delta) * (1-\delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Since $x_k \to x$, there exists $k_1 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda/2) > 1 - \delta,$$
$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda/2) < \delta,$$

and

$$\nu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda/2)<\delta$$

for every $k\geq k_1$.

Similarly, since $x_k\to y$, there exists $k_2\in\mathbb{N}$ such that

$$\mu_F^3(z_1,\dots,z_{n-1},x_k-y,\lambda/2)>1-\delta,$$

$$\eta_F^3(z_1,\dots,z_{n-1},x_k-y,\lambda/2)<\delta,$$

and

$$\nu_F^3(z_1,\dots,z_{n-1},x_k-y,\lambda/2)<\delta$$

for every $k\geq k_2$.

Let

$$k_0=\max\{k_1,k_2\}.$$

Then, all the preceding inequalities hold for every $k\geq k_0$.

Observe that

$$x-y=(x-x_k)+(x_k-y).$$

By the homogeneity condition, with $\alpha=-1$, we have

$$\mu_F(z_1,\dots,z_{n-1},x-x_k,\lambda/2)=\mu_F(z_1,\dots,z_{n-1},x_k-x,\lambda/2).$$

Similarly,

$$\eta_F(z_1,\dots,z_{n-1},x-x_k,\lambda/2)=\eta_F(z_1,\dots,z_{n-1},x_k-x,\lambda/2)$$

and

$$\nu_F(z_1,\dots,z_{n-1},x-x_k,\lambda/2)=\nu_F(z_1,\dots,z_{n-1},x_k-x,\lambda/2).$$

Using the triangle-type condition for μ_F, we obtain

$$\begin{aligned}
&\mu_F^3(z_1,\dots,z_{n-1},x-y,\lambda)\\
&\quad=\mu_F^3\big(z_1,\dots,z_{n-1},(x-x_k)+(x_k-y),\lambda/2+\lambda/2\big)\\
&\quad\geq\mu_F^3(z_1,\dots,z_{n-1},x-x_k,\lambda/2)*\mu_F^3(z_1,\dots,z_{n-1},x_k-y,\lambda/2)\\
&\quad>(1-\delta)*(1-\delta)\\
&\quad>1-\varepsilon.
\end{aligned}$$

Since $\varepsilon\in(0,1)$ is arbitrary, it follows that

$$\mu_F^3(z_1,\dots,z_{n-1},x-y,\lambda)=1.$$

Consequently,

$$\mu_F(z_1,\dots,z_{n-1},x-y,\lambda)=1.$$

Using the triangle-type condition for η_F, we obtain

$$\begin{aligned}
&\eta_F^3(z_1,\dots,z_{n-1},x-y,\lambda)\\
&\quad\leq\eta_F^3(z_1,\dots,z_{n-1},x-x_k,\lambda/2)\diamond\eta_F^3(z_1,\dots,z_{n-1},x_k-y,\lambda/2)\\
&\quad<\delta\diamond\delta\\
&\quad<\varepsilon.
\end{aligned}$$

Since $\varepsilon\in(0,1)$ is arbitrary, it follows that

$$\eta_F^3(z_1,\dots,z_{n-1},x-y,\lambda)=0,$$

and hence

$$\eta_F(z_1,\dots,z_{n-1},x-y,\lambda)=0.$$

Likewise, using the triangle-type condition for ν_F, we obtain

$$\begin{aligned}
&\nu_F^3(z_1,\dots,z_{n-1},x-y,\lambda)\\
&\quad\le \nu_F^3(z_1,\dots,z_{n-1},x-x_k,\lambda/2)\diamond\nu_F^3(z_1,\dots,z_{n-1},x_k-y,\lambda/2)\\
&\quad< \delta\diamond\delta\\
&\quad< \varepsilon.
\end{aligned}$$

Therefore,

$$\nu_F^3(z_1,\dots,z_{n-1},x-y,\lambda)=0,$$

and consequently,

$$\nu_F(z_1,\dots,z_{n-1},x-y,\lambda)=0.$$

By Definition 4.4, any one of the conditions

$$\mu_F(z_1,\dots,z_{n-1},x-y,\lambda)=1,$$

$$\eta_F(z_1,\dots,z_{n-1},x-y,\lambda)=0,$$

or

$$\nu_F(z_1,\dots,z_{n-1},x-y,\lambda)=0$$

implies that

$$z_1,\dots,z_{n-1},x-y$$

are linearly dependent. This contradicts the choice of $z_1,\dots,z_{n-1}$.

Therefore, the assumption $x\neq y$ is false, and hence

$$x=y.$$

Thus, the limit of a convergent sequence in a Fermatean neutrosophic n-normed linear space is unique. □

Theorem 4.3. *Let $\mathcal{A}=(X,\mu_F,\eta_F,\nu_F,*,\diamond)$ be a Fermatean neutrosophic n-normed linear space. Then $x_k\to x$ in $\mathcal{A}$ if and only if, for every $\lambda>0$ and $z_1,\dots,z_{n-1}\in X$,*

$$\mu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)\to 1,$$

$$\eta_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)\to 0,$$

and

$$\nu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda)\to 0$$

as $k\to\infty$.

Proof. Fix $\lambda>0$ and $z_1,\dots,z_{n-1}\in X$. For convenience, write

$$T_k=\mu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda),$$

$$I_k=\eta_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda),$$

and

$$F_k=\nu_F^3(z_1,\dots,z_{n-1},x_k-x,\lambda).$$

Suppose first that

$$x_k\longrightarrow x$$

in the Fermatean neutrosophic n-normed linear space $\mathcal{A}$. By Definition 4.4, for every $\varepsilon\in(0,1)$, there exists $k_0\in\mathbb{N}$ such that

$$T_k>1-\varepsilon,\qquad I_k<\varepsilon,\qquad F_k<\varepsilon$$

for every $k\ge k_0$.

Since $0\le T_k\le 1$, we have

$$|T_k-1|=1-T_k<\varepsilon$$

for every $k \geq k_0$. Therefore,

$$\lim_{k\to\infty} T_k = 1.$$

Similarly, since $0 \leq I_k \leq 1$ and $0 \leq F_k \leq 1$, we have

$$|I_k - 0| = I_k < \varepsilon$$

and

$$|F_k - 0| = F_k < \varepsilon$$

for every $k \geq k_0$. Consequently,

$$\lim_{k\to\infty} I_k = 0 \qquad \text{and} \qquad \lim_{k\to\infty} F_k = 0.$$

Thus,

$$\lim_{k\to\infty} \mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = 1,$$

$$\lim_{k\to\infty} \eta_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = 0,$$

and

$$\lim_{k\to\infty} \nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = 0.$$

Conversely, suppose that, for every $\lambda > 0$ and every $z_1, \ldots, z_{n-1} \in X$,

$$T_k \longrightarrow 1, \qquad I_k \longrightarrow 0, \qquad F_k \longrightarrow 0.$$

Let $\varepsilon \in (0, 1)$. Since $T_k \to 1$, there exists $k_1 \in \mathbb{N}$ such that

$$|T_k - 1| < \varepsilon$$

for every $k \geq k_1$. Hence,

$$T_k > 1 - \varepsilon$$

for every $k \geq k_1$.

Since $I_k \to 0$, there exists $k_2 \in \mathbb{N}$ such that

$$|I_k| < \varepsilon,$$

and hence

$$I_k < \varepsilon$$

for every $k \geq k_2$.

Similarly, since $F_k \to 0$, there exists $k_3 \in \mathbb{N}$ such that

$$|F_k| < \varepsilon,$$

and hence

$$F_k < \varepsilon$$

for every $k \geq k_3$.

Set

$$k_0 = \max\{k_1, k_2, k_3\}.$$

Then, for every $k \geq k_0$,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon,$$

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon.$$

Therefore, by Definition 4.4, we conclude that $x_k \longrightarrow x$. This completes the proof. □

Example 4.5. *Let $X = \mathbb{R}^2$ and define the usual 2-norm on X by $\|u, v\| = \left|\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right|$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are elements of $\mathbb{R}^2$.*

Let

$$a * b = ab$$

be the product t-norm, and let

$$a \diamond b = a + b - ab$$

be the probabilistic-sum t-conorm for all $a, b \in [0, 1]$.

For $u, v \in X$ and $\lambda > 0$, define

$$\mu_F(u, v, \lambda) = \left(\frac{\lambda}{\lambda + \|u, v\|}\right)^{\frac{1}{3}},$$

$$\eta_F(u, v, \lambda) = \left(\frac{\|u, v\|}{\lambda + \|u, v\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(u, v, \lambda) = \left(\frac{\|u, v\|}{\lambda + \|u, v\|}\right)^{\frac{1}{3}}.$$

Thus, in this example,

$$\eta_F(u, v, \lambda) = \nu_F(u, v, \lambda).$$

We have

$$\mu_F^3(u, v, \lambda) + \nu_F^3(u, v, \lambda) = \frac{\lambda}{\lambda + \|u, v\|} + \frac{\|u, v\|}{\lambda + \|u, v\|} = 1.$$

Moreover,

$$\begin{aligned} &\mu_F^3(u, v, \lambda) + \eta_F^3(u, v, \lambda) + \nu_F^3(u, v, \lambda) \\ &\quad = \frac{\lambda + 2\|u, v\|}{\lambda + \|u, v\|} \\ &\quad = 1 + \frac{\|u, v\|}{\lambda + \|u, v\|} \leq 2. \end{aligned}$$

Therefore, the Fermatean neutrosophic admissibility conditions

$$0 \leq \mu_F^3(u, v, \lambda) + \nu_F^3(u, v, \lambda) \leq 1$$

and

$$0 \leq \mu_F^3(u, v, \lambda) + \eta_F^3(u, v, \lambda) + \nu_F^3(u, v, \lambda) \leq 2$$

are satisfied.

Furthermore,

$$\mu_F(u, v, \lambda) = 1$$

if and only if $\|u, v\| = 0$, which is equivalent to u and v being linearly dependent. Similarly,

$$\eta_F(u, v, \lambda) = 0 \quad \text{and} \quad \nu_F(u, v, \lambda) = 0$$

if and only if u and v are linearly dependent. The remaining axioms follow from the corresponding properties of the determinant 2-norm. Hence,

$$\big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$$

is a Fermatean neutrosophic 2-normed linear space.

Now, we define a sequence $\langle x_k \rangle$ in X by

$$x_k = \left(1 + \frac{1}{k}, 2 + \frac{2}{k}\right), \qquad k \in \mathbb{N},$$

and let

$$x = (1, 2).$$

Then,

$$x_k - x = \left(\frac{1}{k}, \frac{2}{k}\right) = \frac{1}{k}(1, 2) = \frac{1}{k}x.$$

Let $z = (z_1, z_2) \in X$ be arbitrary. By the homogeneity of the 2-norm,

$$\|z, x_k - x\| = \left\|z, \frac{1}{k}x\right\| = \frac{1}{k}\|z, x\|.$$

Consequently,

$$\|z, x_k - x\| \longrightarrow 0 \qquad as \qquad k \to \infty.$$

For every $\lambda > 0$, we have

$$\begin{aligned}\mu_F^3(z, x_k - x, \lambda) &= \frac{\lambda}{\lambda + \|z, x_k - x\|} \\ &= \frac{\lambda}{\lambda + \frac{1}{k}\|z, x\|} \longrightarrow 1.\end{aligned}$$

Also,

$$\begin{aligned}\eta_F^3(z, x_k - x, \lambda) &= \frac{\|z, x_k - x\|}{\lambda + \|z, x_k - x\|} \\ &= \frac{\frac{1}{k}\|z, x\|}{\lambda + \frac{1}{k}\|z, x\|} \longrightarrow 0,\end{aligned}$$

and

$$\begin{aligned}\nu_F^3(z, x_k - x, \lambda) &= \frac{\|z, x_k - x\|}{\lambda + \|z, x_k - x\|} \\ &= \frac{\frac{1}{k}\|z, x\|}{\lambda + \frac{1}{k}\|z, x\|} \longrightarrow 0.\end{aligned}$$

Therefore, by Theorem 4.3, $x_k \longrightarrow x$ in the Fermatean neutrosophic 2-normed linear space $\big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$.

5. Cauchy Sequences in Fermatean Neutrosophic n-Normed Linear Spaces(FN-n-NLS)

Cauchy sequences play an important role in the study of convergence and completeness. In classical normed linear spaces, a sequence is Cauchy if the distance between its terms becomes arbitrarily small as the indices become sufficiently large. In a Fermatean neutrosophic n-normed linear space, this idea is expressed through the behavior of the Fermatean truth-membership, indeterminacy-membership, and falsity-membership functions.

More precisely, the closeness between two terms x_k and x_p of a sequence is measured by

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda),$$
$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda),$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda),$$

where $z_1, z_2, \ldots, z_{n-1} \in X$ and $\lambda > 0$. A sequence is expected to be Cauchy whenever the truth-membership degree tends to 1, while the indeterminacy-membership and falsity-membership degrees tend to 0 as $k, p \to \infty$. This agrees with the usual interpretation that the terms of the sequence become increasingly close to one another.

The following definition formalizes the notion of a Cauchy sequence in a Fermatean neutrosophic n-normed linear space.

Definition 5.1. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space. A sequence $\langle x_k \rangle$ in X is called a Cauchy sequence in $\mathcal{A}$ if, for every $\varepsilon \in (0,1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon,$$

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}},$$

$$\eta_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Example 5.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by $\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|$, where $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$.*

Let $a * b = ab$ be the product t-norm, and let $a \diamond b = a + b - ab$ be the probabilistic-sum t-conorm for all $a, b \in [0,1]$.

For $v_1, v_2, \ldots, v_n \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, \ldots, v_n\|} \right)^{\frac{1}{3}},$$

$$\eta_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{2\lambda + \|v_1, v_2, \ldots, v_n\|} \right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{\lambda + \|v_1, v_2, \ldots, v_n\|} \right)^{\frac{1}{3}}.$$

Then,

$$\mu_F^3(v_1, \ldots, v_n, \lambda) + \nu_F^3(v_1, \ldots, v_n, \lambda) = 1.$$

Moreover,

$$\mu_F^3(v_1, \ldots, v_n, \lambda) + \eta_F^3(v_1, \ldots, v_n, \lambda) + \nu_F^3(v_1, \ldots, v_n, \lambda) = 1 + \frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|} \leq 2.$$

Therefore,

$$\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$$

is a Fermatean neutrosophic n-normed linear space.

Now, we fix $u \in X$ and define a sequence $\langle x_k \rangle$ in X by

$$x_k = \frac{u}{k}, \qquad k \in \mathbb{N}.$$

Then $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Proof. Let $\varepsilon \in (0,1)$, $\lambda > 0$, and $z_1, z_2, \ldots, z_{n-1} \in X$ be arbitrary.

For $k, p \in \mathbb{N}$, we have

$$x_k - x_p = \frac{u}{k} - \frac{u}{p} = \left(\frac{1}{k} - \frac{1}{p}\right) u.$$

By the homogeneity property of the usual n-norm,

$$\begin{aligned}&\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| \\ &\quad = \left|\frac{1}{k} - \frac{1}{p}\right| \|z_1, z_2, \ldots, z_{n-1}, u\|.\end{aligned}$$

Put

$$C = \|z_1, z_2, \ldots, z_{n-1}, u\|.$$

Then,

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| = C\left|\frac{1}{k} - \frac{1}{p}\right|.$$

Since

$$\left|\frac{1}{k} - \frac{1}{p}\right| \leq \frac{1}{k} + \frac{1}{p},$$

we obtain, for all $k, p \geq k_0$,

$$\left|\frac{1}{k} - \frac{1}{p}\right| \leq \frac{2}{k_0}.$$

Choose $k_0 \in \mathbb{N}$ sufficiently large so that

$$\frac{2C}{k_0} < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Then, for all $k, p \geq k_0$,

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Now,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = \frac{\lambda}{\lambda + \|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}.$$

Therefore,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > \frac{\lambda}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = 1 - \varepsilon.$$

Furthermore,

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = \frac{\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}{2\lambda + \|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}.$$

Since

$$2\lambda + \|z_1, \ldots, z_{n-1}, x_k - x_p\| > \lambda + \|z_1, \ldots, z_{n-1}, x_k - x_p\|,$$

we have

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \leq \frac{\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}{\lambda + \|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}.$$

Similarly,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = \frac{\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}{\lambda + \|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}.$$

Therefore,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \frac{\frac{\varepsilon\lambda}{1-\varepsilon}}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = \varepsilon.$$

It follows that

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Hence, for all $k, p \geq k_0$,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon,$$
$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}},$$
$$\eta_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}.$$

Therefore, by Definition 5.1, the sequence $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. □

Example 5.2. *Let $X = \mathbb{R}^3$ and define the usual 3-norm on X by $\|v_1, v_2, v_3\| = |\det(v_1, v_2, v_3)|$, where $v_1, v_2, v_3 \in \mathbb{R}^3$.*

Let $a * b = ab$ be the product t-norm, and let $a \diamond b = a + b - ab$ be the probabilistic-sum t-conorm for all $a, b \in [0, 1]$.

For $v_1, v_2, v_3 \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, v_3, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, v_3\|} \right)^{\frac{1}{3}},$$

$$\eta_F(v_1, v_2, v_3, \lambda) = \left(\frac{\|v_1, v_2, v_3\|}{2\lambda + \|v_1, v_2, v_3\|} \right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1, v_2, v_3, \lambda) = \left(\frac{\|v_1, v_2, v_3\|}{\lambda + \|v_1, v_2, v_3\|} \right)^{\frac{1}{3}}.$$

Then,

$$\mu_F^3(v_1, v_2, v_3, \lambda) + \nu_F^3(v_1, v_2, v_3, \lambda) = 1$$

and

$$\mu_F^3(v_1, v_2, v_3, \lambda) + \eta_F^3(v_1, v_2, v_3, \lambda) + \nu_F^3(v_1, v_2, v_3, \lambda) = 1 + \frac{\|v_1, v_2, v_3\|}{2\lambda + \|v_1, v_2, v_3\|} \leq 2.$$

Therefore, $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ is a Fermatean neutrosophic 3-normed linear space.

Define a sequence $\langle x_k \rangle$ in X by

$$x_k = \left(1 + \frac{1}{k}, 2 + \frac{1}{k^2}, 3 + \frac{1}{k^3} \right), \qquad k \in \mathbb{N}.$$

Then $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Proof. Let $\varepsilon \in (0, 1)$, $\lambda > 0$, and $z_1, z_2 \in X$ be arbitrary. We prove that there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) > 1 - \varepsilon,$$
$$\eta_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

For $k, p \in \mathbb{N}$, we have

$$x_k - x_p = \left(\frac{1}{k} - \frac{1}{p}, \frac{1}{k^2} - \frac{1}{p^2}, \frac{1}{k^3} - \frac{1}{p^3} \right).$$

Since the mapping

$$w \longmapsto \det(z_1, z_2, w)$$

is linear in w, there exists a constant $C_{z_1,z_2} \geq 0$, depending only on z_1 and z_2, such that

$$|\det(z_1, z_2, w)| \leq C_{z_1,z_2} \left(|w_1| + |w_2| + |w_3|\right)$$

for every $w = (w_1, w_2, w_3) \in \mathbb{R}^3$.
Hence,

$$\begin{aligned}\|z_1, z_2, x_k - x_p\| &= |\det(z_1, z_2, x_k - x_p)| \\ &\leq C_{z_1,z_2} \left(\left| \frac{1}{k} - \frac{1}{p} \right| + \left| \frac{1}{k^2} - \frac{1}{p^2} \right| + \left| \frac{1}{k^3} - \frac{1}{p^3} \right| \right).\end{aligned}$$

For $k, p \geq k_0$, we have

$$\left| \frac{1}{k} - \frac{1}{p} \right| \leq \frac{2}{k_0},$$

$$\left| \frac{1}{k^2} - \frac{1}{p^2} \right| \leq \frac{2}{k_0^2} \leq \frac{2}{k_0},$$

and

$$\left| \frac{1}{k^3} - \frac{1}{p^3} \right| \leq \frac{2}{k_0^3} \leq \frac{2}{k_0}.$$

Therefore,

$$\|z_1, z_2, x_k - x_p\| \leq \frac{6C_{z_1,z_2}}{k_0}$$

for all $k, p \geq k_0$.

Choose $k_0 \in \mathbb{N}$ sufficiently large so that

$$\frac{6C_{z_1,z_2}}{k_0} < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Then, for all $k, p \geq k_0$,

$$\|z_1, z_2, x_k - x_p\| < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Now,

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) = \frac{\lambda}{\lambda + \|z_1, z_2, x_k - x_p\|}.$$

Thus,

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) > \frac{\lambda}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = 1 - \varepsilon.$$

Furthermore,

$$\eta_F^3(z_1, z_2, x_k - x_p, \lambda) = \frac{\|z_1, z_2, x_k - x_p\|}{2\lambda + \|z_1, z_2, x_k - x_p\|}.$$

Hence,

$$\eta_F^3(z_1, z_2, x_k - x_p, \lambda) \leq \nu_F^3(z_1, z_2, x_k - x_p, \lambda).$$

Similarly,

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) = \frac{\|z_1, z_2, x_k - x_p\|}{\lambda + \|z_1, z_2, x_k - x_p\|}.$$

Therefore,

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) < \frac{\frac{\varepsilon\lambda}{1-\varepsilon}}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = \varepsilon.$$

Consequently,

$$\eta_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon.$$

Hence, for all $k, p \geq k_0$,

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) > 1 - \varepsilon,$$
$$\eta_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon.$$

Equivalently,

$$\mu_F(z_1, z_2, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}},$$
$$\eta_F(z_1, z_2, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, z_2, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}.$$

Therefore, by Definition 5.1, $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. □

Theorem 5.1. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space. If a sequence $\langle x_k \rangle$ in X converges to $x \in X$, then $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.*

Proof. Suppose that $x_k \longrightarrow x$ in the Fermatean neutrosophic n-normed linear space $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$.
We prove that $\langle x_k \rangle$ is a Cauchy sequence.

Let $\varepsilon \in (0, 1)$, $\lambda > 0$, and $z_1, z_2, \ldots, z_{n-1} \in X$ be arbitrary.

Since $*$ is a continuous t-norm and

$$1 * 1 = 1,$$

there exists $\delta_1 \in (0, 1)$ such that

$$(1 - \delta_1) * (1 - \delta_1) > 1 - \varepsilon.$$

Similarly, since $\diamond$ is a continuous t-conorm and

$$0 \diamond 0 = 0,$$

there exists $\delta_2 \in (0, 1)$ such that

$$\delta_2 \diamond \delta_2 < \varepsilon.$$

Set

$$\delta = \min\{\delta_1, \delta_2\}.$$

By the monotonicity of $*$ and $\diamond$, we have

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Since $x_k \to x$, by Definition 4.4, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) > 1 - \delta,$$
$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) < \delta,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) < \delta$$

for every $k \geq k_0$.

Now, let $k, p \geq k_0$. We have

$$x_k - x_p = (x_k - x) + (x - x_p).$$

By the homogeneity condition, with $\alpha = -1$, we obtain

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) = \mu_F(z_1, z_2, \ldots, z_{n-1}, x_p - x, \lambda/2),$$
$$\eta_F(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) = \eta_F(z_1, z_2, \ldots, z_{n-1}, x_p - x, \lambda/2),$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) = \nu_F(z_1, z_2, \ldots, z_{n-1}, x_p - x, \lambda/2).$$

Hence,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) > 1 - \delta,$$
$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) < \delta,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) < \delta.$$

We obtain

$$\begin{aligned}
&\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \\
&\quad = \mu_F^3\big(z_1, z_2, \ldots, z_{n-1}, (x_k - x) + (x - x_p), \lambda/2 + \lambda/2\big) \\
&\quad \geq \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) * \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) \\
&\quad \geq (1 - \delta) * (1 - \delta) \\
&\quad > 1 - \varepsilon.
\end{aligned}$$

Therefore,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon.$$

Next, using the triangle-type condition for η_F, we obtain

$$\begin{aligned}
&\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \\
&\quad = \eta_F^3\big(z_1, z_2, \ldots, z_{n-1}, (x_k - x) + (x - x_p), \lambda/2 + \lambda/2\big) \\
&\quad \leq \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) \diamond \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) \\
&\quad \leq \delta \diamond \delta \\
&\quad < \varepsilon.
\end{aligned}$$

Thus,

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Likewise, using the triangle-type condition for ν_F, we obtain

$$\begin{aligned}
&\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \\
&\quad = \nu_F^3\big(z_1, z_2, \ldots, z_{n-1}, (x_k - x) + (x - x_p), \lambda/2 + \lambda/2\big) \\
&\quad \leq \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) \diamond \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) \\
&\quad \leq \delta \diamond \delta \\
&\quad < \varepsilon.
\end{aligned}$$

Therefore,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Consequently, for every $\varepsilon \in (0, 1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon,$$
$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}},$$
$$\eta_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Hence, by Definition 5.1, $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. □

6. Completeness in Fermatean Neutrosophic n-Normed Linear Spaces(FN-n-NLS)

Completeness is one of the central notions in the study of normed and fuzzy normed structures. In classical analysis, completeness guarantees that every Cauchy sequence has a limit inside the space. This property is essential for developing convergence theory, fixed point results, and approximation methods. In the framework of Fermatean neutrosophic n-normed linear spaces, the concept of completeness is described by using the Fermatean truth-membership, indeterminacy-membership, and falsity-membership functions.

In particular, a Cauchy sequence in this setting is not measured by an ordinary distance only. Instead, it is characterized by the conditions

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 1,$$
$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 0,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 0$$

as $k, p \to \infty$, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$. Thus, completeness means that such a sequence must converge to some element of the underlying linear space X with respect to the Fermatean neutrosophic n-norm structure.

The following definition introduces completeness in a Fermatean neutrosophic n-normed linear space.

Definition 6.1. *A Fermatean neutrosophic n-normed linear space $\mathcal{A} = \left(X, \mu_F, \eta_F, \nu_F, *, \diamond\right)$ is said to be complete if every Cauchy sequence in $\mathcal{A}$ converges to an element of X with respect to the Fermatean neutrosophic n-norm structure.*

Example 6.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm by*

$$\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|.$$

For $\lambda > 0$, define

$$\mu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$
$$\eta_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Let

$$a * b = ab \qquad \text{and} \qquad a \diamond b = a + b - ab.$$

Then,

$$\mu_F^3(v_1, \ldots, v_n, \lambda) + \nu_F^3(v_1, \ldots, v_n, \lambda) = 1,$$

and

$$\begin{aligned}\mu_F^3(v_1, \ldots, v_n, \lambda) + \eta_F^3(v_1, \ldots, v_n, \lambda) + \nu_F^3(v_1, \ldots, v_n, \lambda) \\ = 1 + \frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|} \\ \leq 2.\end{aligned}$$

Therefore, $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ *is a complete Fermatean neutrosophic n-normed linear space.*

Proof. Let $\langle x_k \rangle$ be a Cauchy sequence in $\mathcal{A}$. By Definition 5.1, for every $\varepsilon \in (0,1)$, every $\lambda > 0$, and every $z_1, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon,$$

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

For fixed $z_1, \ldots, z_{n-1} \in X$, put

$$D_{k,p} = \|z_1, \ldots, z_{n-1}, x_k - x_p\|.$$

Since

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x_p, \lambda) = \frac{D_{k,p}}{\lambda + D_{k,p}},$$

the Cauchy condition implies that

$$\frac{D_{k,p}}{\lambda + D_{k,p}} \longrightarrow 0 \qquad \text{as} \qquad k, p \to \infty.$$

Because $\lambda > 0$ is fixed, it follows that

$$D_{k,p} \longrightarrow 0.$$

Hence,

$$\|z_1, \ldots, z_{n-1}, x_k - x_p\| \longrightarrow 0 \qquad \text{as} \qquad k, p \to \infty.$$

Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$. For each $j \in \{1, \ldots, n\}$, choose

$$z_1, \ldots, z_{n-1} = e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n.$$

Then,

$$\|e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n, x_k - x_p\| = |(x_k)_j - (x_p)_j|.$$

Therefore,

$$|(x_k)_j - (x_p)_j| \longrightarrow 0 \qquad \text{as} \qquad k, p \to \infty$$

for every $j \in \{1, \ldots, n\}$. Thus, each coordinate sequence $\langle (x_k)_j \rangle$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, there exists $x_j \in \mathbb{R}$ such that

$$(x_k)_j \longrightarrow x_j \qquad \text{as} \qquad k \to \infty$$

for every $j \in \{1, \ldots, n\}$. Define

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Then,

$$x_k \longrightarrow x$$

in the usual topology of $\mathbb{R}^n$.

For every fixed $z_1, \ldots, z_{n-1} \in X$, the mapping

$$w \longmapsto \det(z_1, \ldots, z_{n-1}, w)$$

is continuous. Therefore,

$$\|z_1, \ldots, z_{n-1}, x_k - x\| \longrightarrow 0.$$

Put

$$D_k = \|z_1, \ldots, z_{n-1}, x_k - x\|.$$

Then, $D_k \to 0$, and consequently,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = \frac{\lambda}{\lambda + D_k} \longrightarrow 1,$$

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = \frac{D_k}{2\lambda + D_k} \longrightarrow 0,$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) = \frac{D_k}{\lambda + D_k} \longrightarrow 0.$$

Thus, by Theorem 4.3, the sequence $\langle x_k \rangle$ converges to x in $\mathcal{A}$. Hence, $\mathcal{A}$ is complete. □

The next lemma gives an equivalent characterization of Cauchy sequences in terms of the limiting behavior of the Fermatean truth-membership, indeterminacy-membership, and falsity-membership functions. This characterization is useful because it allows us to verify Cauchy sequences by checking the limits of μ_F^3, η_F^3, and ν_F^3 directly.

Lemma 6.1. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space. A sequence $\langle x_k \rangle$ in X is Cauchy in $\mathcal{A}$ if and only if, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$,*

$$\lim_{k,p \to \infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 1,$$

and

$$\lim_{k,p \to \infty} \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0,$$

and

$$\lim_{k,p \to \infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Proof. We prove both directions.

First, suppose that $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. By Definition 5.1, for every $\varepsilon \in (0, 1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon,$$

and

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Since

$$0 \leq \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \leq 1,$$

we have

$$\begin{aligned} 0 &\leq 1 - \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \\ &< \varepsilon \end{aligned}$$

for all $k, p \geq k_0$. Hence,

$$\big|\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 1\big| < \varepsilon$$

for all $k, p \geq k_0$. Therefore,

$$\lim_{k,p \to \infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 1.$$

Similarly, since

$$0 \leq \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \leq 1$$

and

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

we have

$$\left|\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 0\right| < \varepsilon$$

for all $k, p \geq k_0$. Thus,

$$\lim_{k,p\to\infty} \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Likewise,

$$0 \leq \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \leq 1$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

we obtain

$$\left|\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 0\right| < \varepsilon$$

for all $k, p \geq k_0$. Therefore,

$$\lim_{k,p\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Conversely, suppose that, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$,

$$\lim_{k,p\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 1,$$

and

$$\lim_{k,p\to\infty} \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0,$$

and

$$\lim_{k,p\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Let $\varepsilon \in (0, 1)$ be arbitrary. Since

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 1$$

as $k, p \to \infty$, there exists $k_1 \in \mathbb{N}$ such that

$$\left|\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 1\right| < \varepsilon$$

for all $k, p \geq k_1$. Hence,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

for all $k, p \geq k_1$.

Since

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 0$$

as $k, p \to \infty$, there exists $k_2 \in \mathbb{N}$ such that

$$\left|\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 0\right| < \varepsilon$$

for all $k, p \geq k_2$. Therefore,

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_2$.

Similarly, since

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 0$$

as $k, p \to \infty$, there exists $k_3 \in \mathbb{N}$ such that

$$\left|\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 0\right| < \varepsilon$$

for all $k, p \geq k_3$. Consequently,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_3$.

Now, take

$$k_0 = \max\{k_1, k_2, k_3\}.$$

Then, for all $k, p \geq k_0$, we have

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon,$$

and

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}},$$

and

$$\eta_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}},$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Therefore, by Definition 5.1, $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. Hence, $\langle x_k \rangle$ is Cauchy in $\mathcal{A}$ if and only if

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 1,$$

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 0,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \longrightarrow 0$$

as $k, p \to \infty$. □

7. Contraction Mappings and Fixed Point Theorem in Fermatean neutrosophic n-normed linear spaces(FN-n-NLS)

Fixed point theory is one of the most important tools in nonlinear analysis. It is widely used to prove the existence and uniqueness of solutions of operator equations, differential equations, integral equations, and iterative processes. In classical metric and normed spaces, contraction mappings play a central role because they guarantee the convergence of successive iterations to a unique fixed point.

In the structure of Fermatean neutrosophic n-normed linear spaces, the notion of contraction must be formulated in terms of the Fermatean truth-membership, indeterminacy-membership, and falsity-membership functions. Instead of measuring the distance between two points only by an ordinary norm, we compare the degrees

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda),$$

and

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda),$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda),$$

where $x, y \in X$, $z_1, z_2, \ldots, z_{n-1} \in X$, and $\lambda > 0$. A contraction mapping should increase the truth-membership degree of closeness and decrease the indeterminacy-membership and falsity-membership degrees of closeness after applying the mapping.

To obtain a more flexible contraction condition, we introduce a triple of control functions (φ, θ, ψ). The function φ controls the truth-membership component, θ controls the indeterminacy-membership component, and ψ controls the falsity-membership component. These

functions are required to preserve the Fermatean neutrosophic cubic conditions so that the controlled degrees remain admissible in the Fermatean neutrosophic sense. The following definition introduces a Fermatean neutrosophic cubic-control function.

Definition 7.1. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space. A triple of functions (φ, θ, ψ) is called a Fermatean neutrosophic cubic-control function if $\varphi, \theta, \psi : [0,1] \to [0,1]$ satisfy the following conditions:*

(1) *φ, θ, and ψ are continuous;*
(2) *φ, θ, and ψ are non-decreasing;*
(3)
$$\varphi(t) \geq 0, \qquad \theta(t) \geq 0, \qquad \psi(t) \geq 0$$
for all $t \in [0,1]$;
(4)
$$\varphi(1) = 1, \qquad \theta(0) = 0, \qquad \psi(0) = 0;$$
(5) *for every $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ and $\lambda > 0$,*
$$\varphi\left(\mu_F(\widetilde{V}, \lambda)\right)^3 + \psi\left(\nu_F(\widetilde{V}, \lambda)\right)^3 \leq 1$$
and
$$\varphi\left(\mu_F(\widetilde{V}, \lambda)\right)^3 + \theta\left(\eta_F(\widetilde{V}, \lambda)\right)^3 + \psi\left(\nu_F(\widetilde{V}, \lambda)\right)^3 \leq 2,$$
whenever
$$\mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 1$$
and
$$\mu_F^3(\widetilde{V}, \lambda) + \eta_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 2.$$

Then, (φ, θ, ψ) is called a Fermatean neutrosophic cubic-control function on $\mathcal{A}$.

The function φ controls the truth-membership degree, θ controls the indeterminacy-membership degree, and ψ controls the falsity-membership degree. The conditions
$$\varphi(\mu_F)^3 + \psi(\nu_F)^3 \leq 1$$
and
$$\varphi(\mu_F)^3 + \theta(\eta_F)^3 + \psi(\nu_F)^3 \leq 2$$
ensure that the controlled triple still satisfies the Fermatean neutrosophic cubic conditions.

After introducing the cubic-control functions, we now define a Fermatean neutrosophic contraction mapping. This type of mapping is designed to control the Fermatean truth-membership, indeterminacy-membership, and falsity-membership degrees between the images of two points under the mapping. The parameters ξ and η allow the contraction conditions to include both deviations from the limiting degrees and the influence of the control functions.

Definition 7.2. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$ be a Fermatean neutrosophic n-normed linear space, and let (φ, θ, ψ) be a Fermatean neutrosophic cubic-control function. Let $\xi, \eta \geq 0$ be fixed constants.*

A mapping $\tau : X \to X$ is called a Fermatean neutrosophic (φ, θ, ψ)-contraction mapping on $\mathcal{A}$ if, for every $x, y \in X$, every $z_1, z_2, \ldots, z_{n-1} \in X$, and every $\lambda > 0$, the following conditions hold:

$$\mu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \geq \frac{\mu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi\left[1 - \mu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)\right] + \eta\varphi\left(\mu_F(z_1, \ldots, z_{n-1}, x - y, \lambda)\right)},$$

and

$$\eta_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \frac{\eta_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi\eta_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda) + \eta\theta\left(\eta_F(z_1, \ldots, z_{n-1}, x - y, \lambda)\right)},$$

and

$$\nu_F^3(z_1,\ldots,z_{n-1},\tau x-\tau y,\lambda)\leq\frac{\nu_F^3(z_1,\ldots,z_{n-1},x-y,\lambda)}{1+\xi\nu_F^3(z_1,\ldots,z_{n-1},x-y,\lambda)+\eta\psi\left(\nu_F(z_1,\ldots,z_{n-1},x-y,\lambda)\right)}.$$

Then τ is called a Fermatean neutrosophic contraction mapping on $\mathcal{A}$.

Example 7.1. *Let $X=\mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1,v_2,\ldots,v_n\|=\left|\det(v_1,v_2,\ldots,v_n)\right|,$$

where $v_1,v_2,\ldots,v_n\in\mathbb{R}^n$.

Let $a*b=ab \quad and \quad a\diamond b=a+b-ab$ for all $a,b\in[0,1]$.

For $v_1,v_2,\ldots,v_n\in X$ and $\lambda>0$, define as

$$\mu_F(v_1,v_2,\ldots,v_n,\lambda)=\left(\frac{\lambda}{\lambda+\|v_1,v_2,\ldots,v_n\|}\right)^{\frac{1}{3}},$$

and

$$\eta_F(v_1,v_2,\ldots,v_n,\lambda)=\left(\frac{\|v_1,v_2,\ldots,v_n\|}{2\lambda+\|v_1,v_2,\ldots,v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1,v_2,\ldots,v_n,\lambda)=\left(\frac{\|v_1,v_2,\ldots,v_n\|}{\lambda+\|v_1,v_2,\ldots,v_n\|}\right)^{\frac{1}{3}}.$$

Then,

$$\mu_F^3(v_1,\ldots,v_n,\lambda)+\nu_F^3(v_1,\ldots,v_n,\lambda)=1,$$

and

$$\begin{aligned}&\mu_F^3(v_1,\ldots,v_n,\lambda)+\eta_F^3(v_1,\ldots,v_n,\lambda)+\nu_F^3(v_1,\ldots,v_n,\lambda)\\&\quad=1+\frac{\|v_1,\ldots,v_n\|}{2\lambda+\|v_1,\ldots,v_n\|}\\&\quad\leq 2.\end{aligned}$$

Therefore, $\mathcal{A}=\big(X,\mu_F,\eta_F,\nu_F,*,\diamond\big)$ is a Fermatean neutrosophic n-normed linear space.

Define

$$\varphi(t)=t^2,\qquad\theta(t)=t^2,\qquad\psi(t)=t^2,\qquad t\in[0,1].$$

Then (φ,θ,ψ) is a Fermatean neutrosophic cubic-control function. Take

$$\xi=1\qquad and\qquad\eta=1.$$

Define a mapping

$$\tau:X\to X$$

by

$$\tau(x)=0$$

for every $x\in X$. Then, τ is a Fermatean neutrosophic (φ,θ,ψ)-contraction mapping on $\mathcal{A}$.

Proof. Let $x,y\in X$, let $z_1,z_2,\ldots,z_{n-1}\in X$, and let $\lambda>0$ be arbitrary. Since

$$\tau(x)=0\qquad\text{and}\qquad\tau(y)=0,$$

we have

$$\tau x-\tau y=0.$$

Therefore,

$$z_1,z_2,\ldots,z_{n-1},\tau x-\tau y=z_1,z_2,\ldots,z_{n-1},0$$

are linearly dependent. Hence,

$$\mu_F(z_1,z_2,\ldots,z_{n-1},\tau x-\tau y,\lambda)=1,$$

and

$$\eta_F(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0,$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0.$$

Consequently,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 1,$$

and

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0.$$

Now, put

$$M = \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda),$$

and

$$I = \eta_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda),$$

and

$$N = \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda).$$

Since $0 \leq M, I, N \leq 1$, we have

$$1 + \xi(1 - M) + \eta\varphi\left(\mu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right) \geq 1.$$

Thus,

$$\frac{M}{1 + \xi(1 - M) + \eta\varphi\left(\mu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)} \leq M \leq 1.$$

Therefore, the truth-membership contraction inequality is satisfied.

Similarly,

$$1 + \xi I + \eta\theta\left(\eta_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right) \geq 1,$$

and hence

$$\frac{I}{1 + \xi I + \eta\theta\left(\eta_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)} \geq 0.$$

Since

$$\eta_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0,$$

the indeterminacy-membership contraction inequality is satisfied.

Likewise,

$$1 + \xi N + \eta\psi\left(\nu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right) \geq 1,$$

and hence

$$\frac{N}{1 + \xi N + \eta\psi\left(\nu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)} \geq 0.$$

Since

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0,$$

the falsity-membership contraction inequality is satisfied.

Therefore, all three contraction inequalities in Definition 7.2 are satisfied. Hence, τ is a Fermatean neutrosophic (φ, θ, ψ)-contraction mapping on $\mathcal{A}$. □

The action of the zero mapping and its effect on the Fermatean neutrosophic cubic degrees are illustrated in Figure 1. The notion of a fixed point in this setting is the same as in classical fixed point theory: a point remains unchanged under the action of the mapping. However, the existence and uniqueness of such a point will be established using the structure of the Fermatean neutrosophic n-normed linear space and the contraction conditions defined above.

(a) Action of the zero mapping (b) Fermatean neutrosophic cubic degrees

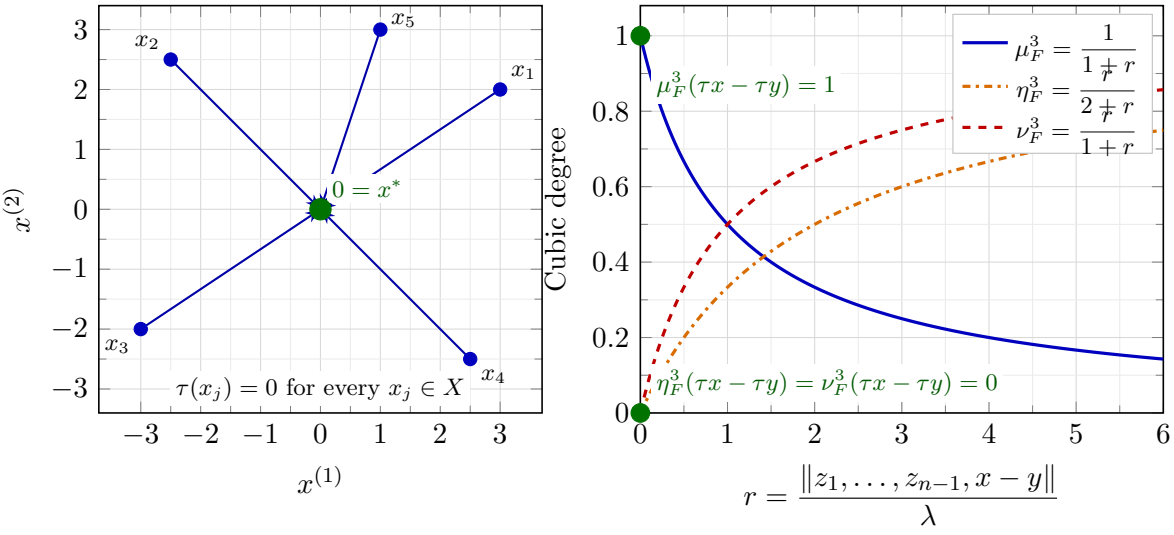

FIGURE 1. The zero contraction mapping: (a) every point is mapped to the unique fixed point $x^* = 0$ in a representative two-dimensional projection, and (b) the cubic truth-membership, indeterminacy-membership, and falsity-membership degrees as functions of the normalized n-norm r. After applying τ, one has $r = 0$ and $(\mu_F^3, \eta_F^3, \nu_F^3) = (1, 0, 0)$.

Definition 7.3. *Let $\mathcal{A} = \left(X, \mu_F, \eta_F, \nu_F, *, \diamond\right)$ be a Fermatean neutrosophic n-normed linear space, and let $\tau : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of τ in $\mathcal{A}$ if $\tau(x) = x$.*

Example 7.2. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|,$$

where $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$.

For $v_1, v_2, \ldots, v_n \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

$$\eta_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{2\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Let

$$a * b = ab \qquad \text{and} \qquad a \diamond b = a + b - ab$$

for all $a, b \in [0, 1]$. Then, $\mathcal{A} = \left(X, \mu_F, \eta_F, \nu_F, *, \diamond\right)$ is a Fermatean neutrosophic n-normed linear space.

Define a mapping

$$\tau : X \to X$$

by

$$\tau(x) = c + \frac{1}{2}x,$$

where

$$c = (1, 2, 3, \ldots, n) \in \mathbb{R}^n.$$

We claim that $x^* = 2c = (2, 4, 6, \ldots, 2n)$ *is a fixed point of* τ *in* $\mathcal{A}$.

Proof. By Definition 7.3, a point $x^* \in X$ is a fixed point of τ if

$$\tau(x^*) = x^*.$$

Take

$$x^* = 2c.$$

Then,

$$\tau(x^*) = c + \frac{1}{2}x^*.$$

Since $x^* = 2c$, we obtain

$$\tau(x^*) = c + \frac{1}{2}(2c) = c + c = 2c.$$

Thus,

$$\tau(x^*) = x^*.$$

Hence,

$$x^* = 2c = (2, 4, 6, \ldots, 2n)$$

is a fixed point of τ in the Fermatean neutrosophic n-normed linear space $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, *, \diamond\big)$. Moreover, if $x \in X$ is any fixed point of τ, then

$$\tau(x) = x.$$

Therefore,

$$c + \frac{1}{2}x = x,$$

which implies

$$c = \frac{1}{2}x.$$

Thus,

$$x = 2c.$$

Therefore, the fixed point of τ is unique. □

(a) Component profiles for $n = 8$ (b) Geometric convergence

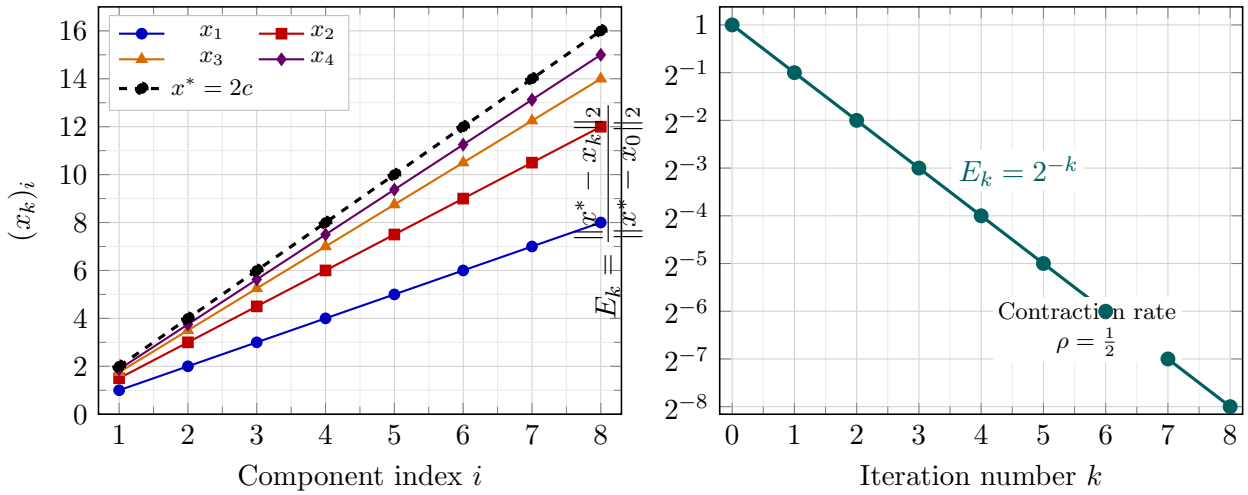

FIGURE 2. Fixed point iteration for $\tau(x) = c + \frac{1}{2}x$: (a) componentwise convergence to $x^* = 2c$ in the representative case $n = 8$, and (b) geometric decay of the normalized error with contraction rate $\rho = \frac{1}{2}$.

The following theorem is a Banach-type fixed point theorem in a complete Fermatean neutrosophic n-normed linear space. It shows that if a mapping satisfies an appropriate Fermatean neutrosophic contraction condition, then the Picard iteration

$$x_{k+1} = \tau x_k$$

converges to a unique fixed point of τ. This result provides the foundation for applying the proposed space to operator equations and related problems.

Theorem 7.1. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, \min, \max\big)$ be a complete Fermatean neutrosophic n-normed linear space. Assume that, for every fixed $\widetilde{V} \in X^n$, $\mu_F(\widetilde{V}, \lambda)$ is non-decreasing in λ, whereas $\eta_F(\widetilde{V}, \lambda)$ and $\nu_F(\widetilde{V}, \lambda)$ are non-increasing in λ, and*

$$\lim_{\lambda\to\infty} \mu_F(\widetilde{V}, \lambda) = 1, \ \lim_{\lambda\to\infty} \eta_F(\widetilde{V}, \lambda) = 0,$$

and

$$\lim_{\lambda\to\infty} \nu_F(\widetilde{V}, \lambda) = 0.$$

Suppose that $\tau : X \to X$ is a Fermatean neutrosophic n-contraction mapping; that is, there exists a constant $\rho \in (0,1)$ such that, for every $x, y \in X$, every $z_1, z_2, \ldots, z_{n-1} \in X$, and every $\lambda > 0$,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \geq \mu_F^3\left(z_1, z_2, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right),$$

$$\eta_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \eta_F^3\left(z_1, z_2, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right),$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \nu_F^3\left(z_1, z_2, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right).$$

Then τ has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\langle x_k \rangle$ in X by

$$x_{k+1} = \tau x_k, \qquad k = 0, 1, 2, \ldots.$$

We first prove that $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Let $\varepsilon \in (0,1)$, $\lambda > 0$, and $z_1, z_2, \ldots, z_{n-1} \in X$ be arbitrary. For $j \geq 1$, since

$$x_{j+1} - x_j = \tau x_j - \tau x_{j-1},$$

the contraction conditions give

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_j - x_{j-1}, \frac{\lambda}{\rho}\right),$$

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, \lambda) \leq \eta_F^3\left(z_1, \ldots, z_{n-1}, x_j - x_{j-1}, \frac{\lambda}{\rho}\right),$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, \lambda) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_j - x_{j-1}, \frac{\lambda}{\rho}\right).$$

For every $j \geq 1$ and every $t > 0$, we obtain

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t}{\rho^j}\right),$$

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t) \leq \eta_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t}{\rho^j}\right),$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t}{\rho^j}\right).$$

Now, let $p > k$. Then,

$$x_p - x_k = (x_p - x_{p-1}) + (x_{p-1} - x_{p-2}) + \cdots + (x_{k+1} - x_k).$$

For $j = k, k+1, \ldots, p-1$, put

$$t_j = (1-\rho)\rho^{j-k}\lambda.$$

Then

$$\begin{aligned}\sum_{j=k}^{p-1} t_j &= (1-\rho)\lambda \sum_{j=k}^{p-1} \rho^{j-k} \\ &= \lambda\left(1-\rho^{p-k}\right) < \lambda.\end{aligned}$$

Since μ_F is non-decreasing with respect to λ, and since the t-norm is min, the triangle-type condition gives

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \geq \min_{k \leq j \leq p-1} \mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j).$$

Using the estimate for successive differences, we obtain

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t_j}{\rho^j}\right).$$

But

$$\frac{t_j}{\rho^j} = \frac{(1-\rho)\rho^{j-k}\lambda}{\rho^j} = (1-\rho)\lambda\rho^{-k}.$$

Hence,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right).$$

Since $\rho \in (0,1)$,

$$(1-\rho)\lambda\rho^{-k} \longrightarrow \infty \qquad \text{as} \qquad k \to \infty.$$

Therefore,

$$\begin{aligned}&\mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right) \\ &\qquad \longrightarrow 1.\end{aligned}$$

Thus, there exists $k_1 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) > 1 - \varepsilon$$

for all $p > k \geq k_1$.

Similarly, since η_F is non-increasing with respect to λ and the t-conorm is max, the triangle-type condition gives

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \leq \max_{k \leq j \leq p-1} \eta_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j).$$

Using the contraction condition repeatedly, we get

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j) \leq \eta_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right).$$

Therefore,

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \leq \eta_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right).$$

Since

$$(1-\rho)\lambda\rho^{-k} \longrightarrow \infty,$$

we obtain

$$\eta_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right) \longrightarrow 0.$$

Hence, there exists $k_2 \in \mathbb{N}$ such that

$$\eta_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) < \varepsilon$$

for all $p > k \geq k_2$.

Likewise, since ν_F is non-increasing with respect to λ, the triangle-type condition gives

$$\nu_F^3(z_1,\ldots,z_{n-1},x_p-x_k,\lambda) \le \max_{k\le j\le p-1} \nu_F^3(z_1,\ldots,z_{n-1},x_{j+1}-x_j,t_j).$$

Using the contraction condition repeatedly, we get

$$\nu_F^3(z_1,\ldots,z_{n-1},x_{j+1}-x_j,t_j) \le \nu_F^3\left(z_1,\ldots,z_{n-1},x_1-x_0,(1-\rho)\lambda\rho^{-k}\right).$$

Therefore,

$$\nu_F^3(z_1,\ldots,z_{n-1},x_p-x_k,\lambda) \le \nu_F^3\left(z_1,\ldots,z_{n-1},x_1-x_0,(1-\rho)\lambda\rho^{-k}\right).$$

Again, since

$$(1-\rho)\lambda\rho^{-k} \longrightarrow \infty,$$

we obtain

$$\nu_F^3\left(z_1,\ldots,z_{n-1},x_1-x_0,(1-\rho)\lambda\rho^{-k}\right) \longrightarrow 0.$$

Hence, there exists $k_3 \in \mathbb{N}$ such that

$$\nu_F^3(z_1,\ldots,z_{n-1},x_p-x_k,\lambda) < \varepsilon$$

for all $p > k \ge k_3$.

Let

$$k_0 = \max\{k_1,k_2,k_3\}.$$

Then, for all $p > k \ge k_0$,

$$\mu_F^3(z_1,\ldots,z_{n-1},x_p-x_k,\lambda) > 1-\varepsilon,$$
$$\eta_F^3(z_1,\ldots,z_{n-1},x_p-x_k,\lambda) < \varepsilon,$$

and

$$\nu_F^3(z_1,\ldots,z_{n-1},x_p-x_k,\lambda) < \varepsilon.$$

If $k > p$, the same inequalities follow by interchanging k and p and using the homogeneity condition with scalar -1; if $k = p$, they hold trivially. Therefore, by Definition 5.1, the sequence $\langle x_k\rangle$ is Cauchy in $\mathcal{A}$.

Since $\mathcal{A}$ is complete, there exists $u \in X$ such that $x_k \longrightarrow u$ in $\mathcal{A}$.
We now prove that u is a fixed point of τ. Since

$$x_{k+1} = \tau x_k,$$

we have

$$x_{k+1} - \tau u = \tau x_k - \tau u.$$

Applying the contraction conditions, for every $\lambda > 0$,

$$\mu_F^3(z_1,\ldots,z_{n-1},x_{k+1}-\tau u,\lambda) \ge \mu_F^3\left(z_1,\ldots,z_{n-1},x_k-u,\frac{\lambda}{\rho}\right),$$

$$\eta_F^3(z_1,\ldots,z_{n-1},x_{k+1}-\tau u,\lambda) \le \eta_F^3\left(z_1,\ldots,z_{n-1},x_k-u,\frac{\lambda}{\rho}\right),$$

and

$$\nu_F^3(z_1,\ldots,z_{n-1},x_{k+1}-\tau u,\lambda) \le \nu_F^3\left(z_1,\ldots,z_{n-1},x_k-u,\frac{\lambda}{\rho}\right).$$

Since $x_k \to u$, the right-hand sides tend to 1, 0, and 0, respectively. Hence,

$$\mu_F^3(z_1,\ldots,z_{n-1},x_{k+1}-\tau u,\lambda) \longrightarrow 1,$$
$$\eta_F^3(z_1,\ldots,z_{n-1},x_{k+1}-\tau u,\lambda) \longrightarrow 0,$$

and

$$\nu_F^3(z_1,\ldots,z_{n-1},x_{k+1}-\tau u,\lambda) \longrightarrow 0.$$

Therefore,

$$x_{k+1} \longrightarrow \tau u.$$

On the other hand, since $x_k \to u$, the shifted sequence $\langle x_{k+1} \rangle$ also converges to u. By the uniqueness of the limit in Theorem 4.2, we obtain

$$\tau u = u.$$

Thus, u is a fixed point of τ.

It remains to prove uniqueness. Suppose that $u, v \in X$ are fixed points of τ. Then

$$\tau u = u \qquad \text{and} \qquad \tau v = v.$$

For every $\lambda > 0$, the truth-membership contraction condition gives

$$\begin{aligned} &\mu_F^3(z_1, \ldots, z_{n-1}, u - v, \lambda) \\ &\quad = \mu_F^3(z_1, \ldots, z_{n-1}, \tau u - \tau v, \lambda) \\ &\quad \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, u - v, \frac{\lambda}{\rho}\right). \end{aligned}$$

Applying this repeatedly, we obtain

$$\begin{aligned} &\mu_F^3(z_1, \ldots, z_{n-1}, u - v, \lambda) \\ &\quad \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, u - v, \frac{\lambda}{\rho^m}\right) \end{aligned}$$

for every $m \in \mathbb{N}$. Since $\rho \in (0, 1)$,

$$\frac{\lambda}{\rho^m} \longrightarrow \infty \qquad \text{as} \qquad m \to \infty.$$

Hence,

$$\mu_F^3\left(z_1, \ldots, z_{n-1}, u - v, \frac{\lambda}{\rho^m}\right) \longrightarrow 1.$$

Therefore,

$$\mu_F^3(z_1, \ldots, z_{n-1}, u - v, \lambda) = 1,$$

and thus

$$\mu_F(z_1, \ldots, z_{n-1}, u - v, \lambda) = 1.$$

This means that

$$z_1, z_2, \ldots, z_{n-1}, u - v$$

are linearly dependent.

If $u \neq v$, then $u - v \neq 0$. We may choose $z_1, z_2, \ldots, z_{n-1} \in X$ such that

$$z_1, z_2, \ldots, z_{n-1}, u - v$$

are linearly independent. This contradicts the conclusion above. Hence, $u = v$. Therefore, τ has a unique fixed point in X. □

8. Application

In this section, we present an application of the fixed point theorem in a Fermatean neutrosophic n-normed linear space. The main purpose is to show that the abstract fixed point result can be used to guarantee the existence and uniqueness of solutions of operator equations. In particular, we consider an affine operator equation of the form

$$x = c + \alpha x,$$

where $c \in X$ is fixed and α is a scalar satisfying $|\alpha| < 1$.

This equation can be rewritten as the fixed point problem

$$\tau(x) = x,$$

where $\tau : X \to X$ is defined by

$$\tau(x) = c + \alpha x.$$

Therefore, if τ satisfies the Fermatean neutrosophic contraction condition, then the fixed point theorem ensures that the equation has a unique solution.

Theorem 8.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by $\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|$. For $v_1, \ldots, v_n \in X$ and $\lambda > 0$, define*

$$\mu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

$$\eta_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Let

$$a * b = \min\{a, b\} \qquad \text{and} \qquad a \diamond b = \max\{a, b\}.$$

Then, $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, \min, \max\big)$ is a complete Fermatean neutrosophic n-normed linear space.

Let $c \in X$ be fixed and let $\alpha \in \mathbb{R}$ satisfy $|\alpha| < 1$. Consider the operator equation $x = c + \alpha x$. Then this equation has a unique solution in X.

Proof. First, if

$$D = \|v_1, \ldots, v_n\|,$$

then

$$\mu_F^3(v_1, \ldots, v_n, \lambda) + \nu_F^3(v_1, \ldots, v_n, \lambda) = \frac{\lambda}{\lambda + D} + \frac{D}{\lambda + D} = 1.$$

Moreover,

$$\mu_F^3(v_1, \ldots, v_n, \lambda) + \eta_F^3(v_1, \ldots, v_n, \lambda) + \nu_F^3(v_1, \ldots, v_n, \lambda) = 1 + \frac{D}{2\lambda + D} \leq 2.$$

Thus, the Fermatean neutrosophic admissibility conditions are satisfied.

Define a mapping

$$\tau : X \to X$$

by

$$\tau(x) = c + \alpha x, \qquad x \in X.$$

The operator equation

$$x = c + \alpha x$$

is equivalent to the fixed point equation

$$\tau(x) = x.$$

Choose $\rho \in (0, 1)$ such that

$$|\alpha| \leq \rho < 1.$$

Let $x, y \in X$, let $z_1, z_2, \ldots, z_{n-1} \in X$, and let $\lambda > 0$. Since

$$\tau x - \tau y = (c + \alpha x) - (c + \alpha y) = \alpha(x - y),$$

we have

$$\|z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y\| = |\alpha| \, \|z_1, z_2, \ldots, z_{n-1}, x - y\|.$$

Put

$$D = \|z_1, z_2, \ldots, z_{n-1}, x - y\|.$$

Then,

$$\|z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y\| = |\alpha| D \leq \rho D.$$

For the truth-membership function, we obtain

$$\mu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = \frac{\lambda}{\lambda + |\alpha| D}.$$

Because $|\alpha| \leq \rho$, it follows that

$$\frac{\lambda}{\lambda + |\alpha| D} \geq \frac{\lambda}{\lambda + \rho D}.$$

But

$$\frac{\lambda}{\lambda + \rho D} = \frac{\lambda/\rho}{\lambda/\rho + D}.$$

Therefore,

$$\mu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right).$$

For the indeterminacy-membership function, we have

$$\eta_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = \frac{|\alpha| D}{2\lambda + |\alpha| D}.$$

The function

$$r \mapsto \frac{rD}{2\lambda + rD}$$

is non-decreasing for $r \geq 0$. Consequently,

$$\frac{|\alpha| D}{2\lambda + |\alpha| D} \leq \frac{\rho D}{2\lambda + \rho D}.$$

Since

$$\frac{\rho D}{2\lambda + \rho D} = \frac{D}{2\lambda/\rho + D},$$

we obtain

$$\eta_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \eta_F^3\left(z_1, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right).$$

Similarly, for the falsity-membership function,

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = \frac{|\alpha| D}{\lambda + |\alpha| D}.$$

Therefore,

$$\frac{|\alpha| D}{\lambda + |\alpha| D} \leq \frac{\rho D}{\lambda + \rho D} = \frac{D}{\lambda/\rho + D}.$$

Hence,

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right).$$

Thus, τ is a Fermatean neutrosophic n-contraction mapping with contraction constant $\rho \in (0, 1)$. Since $\mathcal{A}$ is complete, Theorem **??** implies that τ has a unique fixed point $x^* \in X$. Therefore,

$$\tau(x^*) = x^*.$$

Hence,

$$c + \alpha x^* = x^*,$$

and consequently,

$$(1 - \alpha) x^* = c.$$

Since $|\alpha| < 1$, we have $\alpha \neq 1$. Thus,

$$x^* = \frac{c}{1-\alpha}.$$

Therefore, the equation

$$x = c + \alpha x$$

has the unique solution

$$x^* = \frac{c}{1-\alpha}.$$

□

This application shows that the fixed point theorem in a Fermatean neutrosophic n-normed linear space can be used to guarantee the existence and uniqueness of solutions of operator equations of the form

$$x = \tau(x).$$

In particular, the affine equation

$$x = c + \alpha x$$

has a unique solution whenever $|\alpha| < 1$.

Example 8.1. *Let $X = \mathbb{R}^2$ and define the usual 2-norm on X by*

$$\|u, v\| = \left| \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \right|,$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are elements of $\mathbb{R}^2$.

For $u, v \in X$ and $\lambda > 0$, define

$$\mu_F(u, v, \lambda) = \left(\frac{\lambda}{\lambda + \|u, v\|} \right)^{\frac{1}{3}},$$

$$\eta_F(u, v, \lambda) = \left(\frac{\|u, v\|}{2\lambda + \|u, v\|} \right)^{\frac{1}{3}},$$

and

$$\nu_F(u, v, \lambda) = \left(\frac{\|u, v\|}{\lambda + \|u, v\|} \right)^{\frac{1}{3}}.$$

Let

$$a * b = \min\{a, b\}, \qquad a \diamond b = \max\{a, b\},$$

for all $a, b \in [0, 1]$. Then $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, \min, \max\big)$ is a complete Fermatean neutrosophic 2-normed linear space.

Define the mapping

$$\tau : X \to X$$

by

$$\tau(x) = c + \alpha x,$$

where

$$c = (2, 4) \qquad \text{and} \qquad \alpha = \frac{1}{2}.$$

Thus,

$$\tau(x) = (2, 4) + \frac{1}{2}x.$$

Since

$$|\alpha| = \frac{1}{2} < 1,$$

the mapping τ is a Fermatean neutrosophic contraction mapping with contraction constant

$$\rho = \frac{1}{2}.$$

The fixed point x^* of τ satisfies

$$\tau(x^*) = x^*.$$

Hence,

$$(2,4) + \frac{1}{2}x^* = x^*.$$

Therefore,

$$\frac{1}{2}x^* = (2,4),$$

and so

$$x^* = (4,8).$$

Thus, the unique fixed point of τ is

$$x^* = (4,8).$$

Indeed,

$$\tau(4,8) = (2,4) + \frac{1}{2}(4,8) = (2,4) + (2,4) = (4,8).$$

Therefore,

$$\tau(x^*) = x^*.$$

Now, we verify the contraction behavior numerically. Take

$$x = (0,0), \qquad y = (2,1), \qquad z = (1,3), \qquad \lambda = 2.$$

Then,

$$x - y = (-2,-1).$$

Hence,

$$\|z, x-y\| = \left|\det \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}\right| = |-1+6| = 5.$$

Therefore,

$$\mu_F^3(z, x-y, 2) = \frac{2}{2+5} = \frac{2}{7},$$
$$\eta_F^3(z, x-y, 2) = \frac{5}{2(2)+5} = \frac{5}{9},$$

and

$$\nu_F^3(z, x-y, 2) = \frac{5}{2+5} = \frac{5}{7}.$$

Next,

$$\tau(x) = (2,4) + \frac{1}{2}(0,0) = (2,4),$$

and

$$\tau(y) = (2,4) + \frac{1}{2}(2,1) = \left(3, \frac{9}{2}\right).$$

Thus,

$$\tau(x) - \tau(y) = (2,4) - \left(3, \frac{9}{2}\right) = \left(-1, -\frac{1}{2}\right).$$

Hence,

$$\|z, \tau(x) - \tau(y)\| = \left|\det \begin{pmatrix} 1 & -1 \\ 3 & -\frac{1}{2} \end{pmatrix}\right| = \left|-\frac{1}{2} + 3\right| = \frac{5}{2}.$$

Therefore,

$$\mu_F^3(z, \tau(x) - \tau(y), 2) = \frac{2}{2+\frac{5}{2}} = \frac{4}{9},$$
$$\eta_F^3(z, \tau(x) - \tau(y), 2) = \frac{\frac{5}{2}}{2(2)+\frac{5}{2}} = \frac{5}{13},$$

and

$$\nu_F^3(z, \tau(x) - \tau(y), 2) = \frac{\frac{5}{2}}{2 + \frac{5}{2}} = \frac{5}{9}.$$

Now, we observe that

$$\frac{4}{9} > \frac{2}{7},$$

so

$$\mu_F^3(z, \tau(x) - \tau(y), 2) > \mu_F^3(z, x - y, 2).$$

Moreover,

$$\frac{5}{13} < \frac{5}{9},$$

so

$$\eta_F^3(z, \tau(x) - \tau(y), 2) < \eta_F^3(z, x - y, 2).$$

Also,

$$\frac{5}{9} < \frac{5}{7},$$

so

$$\nu_F^3(z, \tau(x) - \tau(y), 2) < \nu_F^3(z, x - y, 2).$$

Thus, under the mapping τ, the truth-membership degree increases, while the indeterminacy-membership and falsity-membership degrees decrease. This confirms the contraction behavior in the Fermatean neutrosophic 2-normed linear space.

Finally, starting from

$$x_0 = (0, 0),$$

the fixed point iteration

$$x_{k+1} = \tau(x_k)$$

gives

$$x_1 = \tau(x_0) = (2, 4),$$

$$x_2 = \tau(x_1) = (3, 6),$$

$$x_3 = \tau(x_2) = \left(\frac{7}{2}, 7\right),$$

$$x_4 = \tau(x_3) = \left(\frac{15}{4}, \frac{15}{2}\right).$$

In general,

$$x_k = \left(4\left(1 - \frac{1}{2^k}\right), 8\left(1 - \frac{1}{2^k}\right)\right),$$

and hence

$$x_k \longrightarrow (4, 8) \qquad as \qquad k \to \infty.$$

Therefore, τ has the unique fixed point $x^* = (4, 8)$.

The convergence of the fixed point iteration is illustrated in Figure 3. The iterates remain on the line $x^{(2)} = 2x^{(1)}$ and approach the unique fixed point $(4, 8)$.

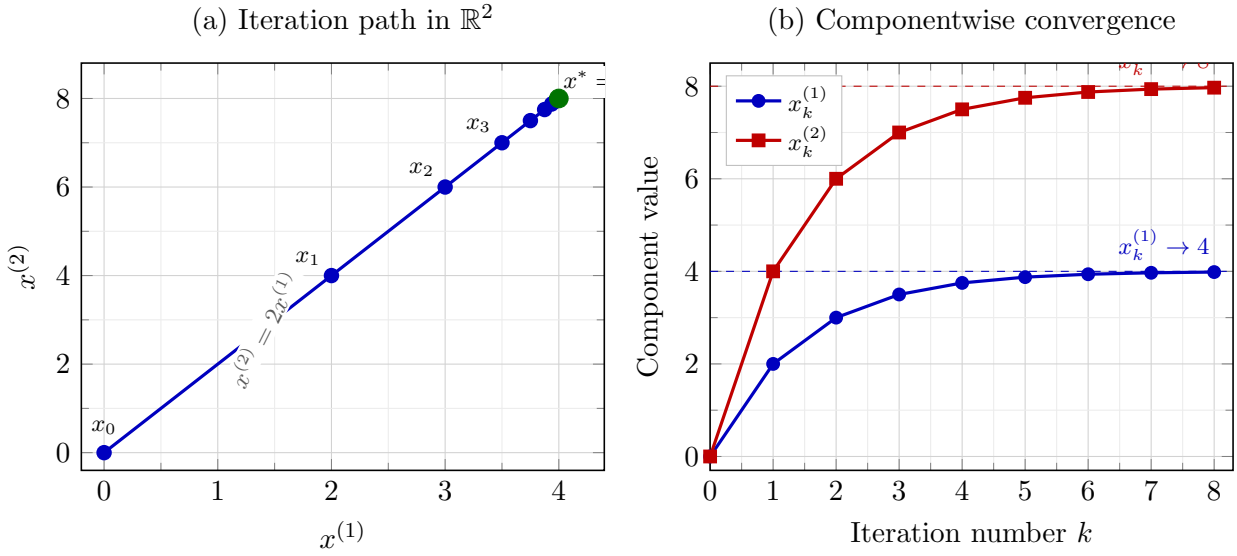

FIGURE 3. Fixed point iteration for $\tau(x) = (2, 4) + \frac{1}{2}x$ starting from $x_0 = (0, 0)$: (a) the iteration path in $\mathbb{R}^2$ and (b) the componentwise convergence to $x^* = (4, 8)$.

8.1. **Application to a Nonlinear Operator Equation.** We now apply the fixed point theorem to establish the existence and uniqueness of a solution of the nonlinear operator equation

$$x = c + G(x),$$

where $c \in X$ is fixed and $G : X \to X$ is a nonlinear operator.

Theorem 8.2. *Let $\mathcal{A} = \big(X, \mu_F, \eta_F, \nu_F, \min, \max\big)$ be a complete Fermatean neutrosophic n-normed linear space. Suppose that $G : X \to X$ satisfies*

$$\|z_1, z_2, \ldots, z_{n-1}, G(x) - G(y)\| \leq q\|z_1, z_2, \ldots, z_{n-1}, x - y\|$$

for every $x, y \in X$, every $z_1, z_2, \ldots, z_{n-1} \in X$, and some constant

$$0 < q < 1.$$

Assume that the Fermatean neutrosophic functions are defined by

$$\mu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

$$\eta_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Then, for every fixed $c \in X$, the nonlinear operator equation

$$x = c + G(x)$$

has a unique solution in X.

Proof. Define the mapping

$$\tau : X \to X$$

by

$$\tau(x) = c + G(x).$$

Then,

$$\tau(x) - \tau(y) = G(x) - G(y).$$

Therefore,

$$\|z_1, \ldots, z_{n-1}, \tau(x) - \tau(y)\| \leq q\|z_1, \ldots, z_{n-1}, x - y\|.$$

Put

$$D = \|z_1, \ldots, z_{n-1}, x - y\|$$

and

$$D_\tau = \|z_1, \ldots, z_{n-1}, \tau(x) - \tau(y)\|.$$

Then,

$$D_\tau \leq qD.$$

For the truth-membership function, we have

$$\mu_F^3(z_1, \ldots, z_{n-1}, \tau(x) - \tau(y), \lambda) = \frac{\lambda}{\lambda + D_\tau}.$$

Since $D_\tau \leq qD$, we obtain

$$\frac{\lambda}{\lambda + D_\tau} \geq \frac{\lambda}{\lambda + qD}.$$

Moreover,

$$\frac{\lambda}{\lambda + qD} = \frac{\lambda/q}{\lambda/q + D}.$$

Consequently,

$$\mu_F^3(z_1, \ldots, z_{n-1}, \tau(x) - \tau(y), \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x - y, \frac{\lambda}{q}\right).$$

For the indeterminacy-membership function, we have

$$\eta_F^3(z_1, \ldots, z_{n-1}, \tau(x) - \tau(y), \lambda) = \frac{D_\tau}{2\lambda + D_\tau}.$$

Therefore,

$$\frac{D_\tau}{2\lambda + D_\tau} \leq \frac{qD}{2\lambda + qD} = \frac{D}{2\lambda/q + D}.$$

Hence,

$$\eta_F^3(z_1, \ldots, z_{n-1}, \tau(x) - \tau(y), \lambda) \leq \eta_F^3\left(z_1, \ldots, z_{n-1}, x - y, \frac{\lambda}{q}\right).$$

Similarly, for the falsity-membership function,

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau(x) - \tau(y), \lambda) = \frac{D_\tau}{\lambda + D_\tau}.$$

Thus,

$$\frac{D_\tau}{\lambda + D_\tau} \leq \frac{qD}{\lambda + qD} = \frac{D}{\lambda/q + D}.$$

Therefore,

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau(x) - \tau(y), \lambda) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x - y, \frac{\lambda}{q}\right).$$

Hence, τ is a Fermatean neutrosophic n-contraction mapping with contraction constant q. Since $\mathcal{A}$ is complete, Theorem **??** implies that τ has a unique fixed point $x^* \in X$. Thus,

$$\tau(x^*) = x^*.$$

Equivalently,

$$x^* = c + G(x^*).$$

Therefore, the nonlinear operator equation

$$x = c + G(x)$$

has a unique solution in X. □

Example 8.2. *Let $X = \mathbb{R}$ with the usual 1-norm $\|x\| = |x|$. For $x \in \mathbb{R}$ and $\lambda > 0$, define*

$$\mu_F(x,\lambda) = \left(\frac{\lambda}{\lambda + |x|}\right)^{\frac{1}{3}},$$

$$\eta_F(x,\lambda) = \left(\frac{|x|}{2\lambda + |x|}\right)^{\frac{1}{3}},$$

and

$$\nu_F(x,\lambda) = \left(\frac{|x|}{\lambda + |x|}\right)^{\frac{1}{3}}.$$

Let

$$a * b = \min\{a,b\}, \qquad a \diamond b = \max\{a,b\}.$$

Then $\mathcal{A} = \big(\mathbb{R}, \mu_F, \eta_F, \nu_F, \min, \max\big)$ is a complete Fermatean neutrosophic 1-normed linear space.

Consider the nonlinear equation

$$x = 1 + \frac{1}{2}\tanh(x-1).$$

Define

$$\tau(x) = 1 + \frac{1}{2}\tanh(x-1).$$

For every $x, y \in \mathbb{R}$, the mean value theorem and the inequality

$$|\tanh u - \tanh v| \leq |u - v|$$

give

$$|\tau(x) - \tau(y)| = \frac{1}{2}|\tanh(x-1) - \tanh(y-1)| \leq \frac{1}{2}|x-y|.$$

Thus, τ is a contraction with contraction constant

$$q = \frac{1}{2}.$$

Therefore, by Theorem 8.2, the nonlinear equation has a unique solution.

We observe that

$$\tau(1) = 1 + \frac{1}{2}\tanh(0) = 1.$$

Hence, the unique solution is

$$x^* = 1.$$

Starting with $x_0 = 0$, the fixed point iteration

$$x_{k+1} = 1 + \frac{1}{2}\tanh(x_k - 1)$$

gives approximately

$$x_1 = 0.619203,$$
$$x_2 = 0.818310,$$
$$x_3 = 0.910140,$$
$$x_4 = 0.955190.$$

Continuing this process gives

$$x_k \longrightarrow 1.$$

Thus, the fixed point iteration converges to the unique solution

$$x^* = 1.$$

The convergence of the fixed point iteration is illustrated in Figure 4. The unique intersection of $y = x$ and $y = \tau(x)$ occurs at $(1, 1)$.

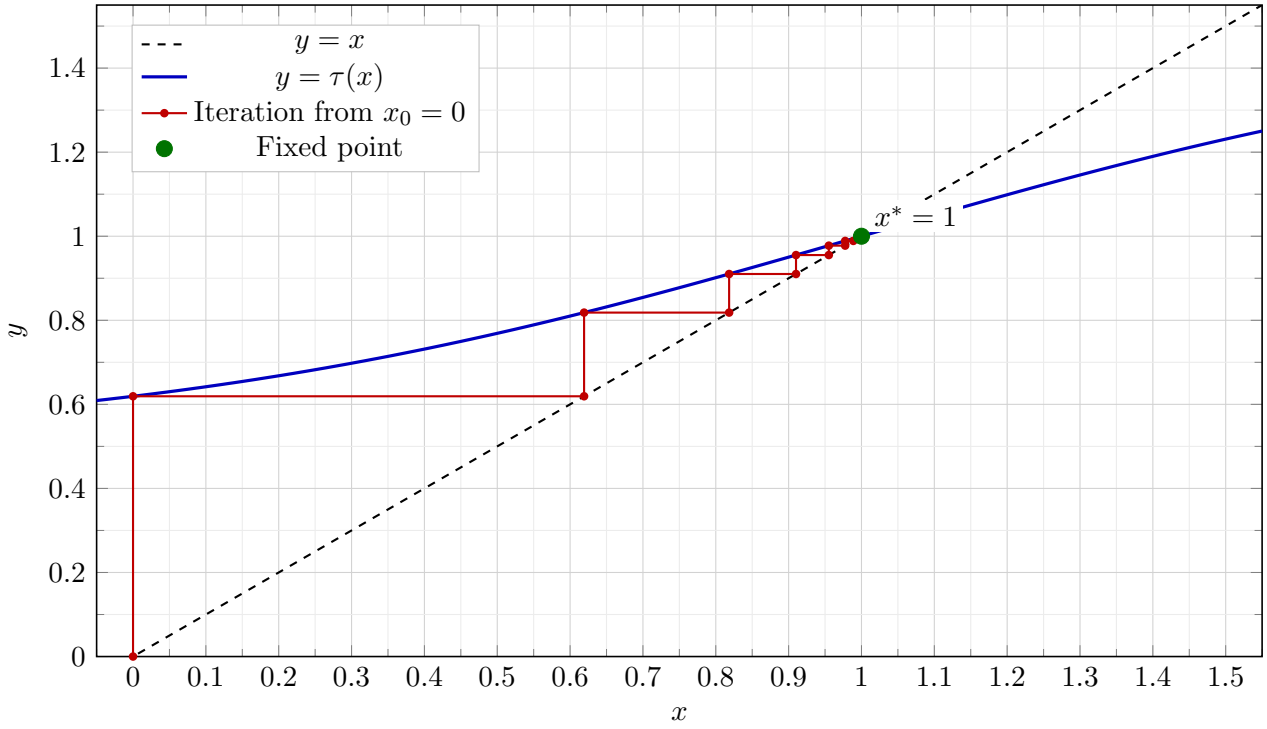

FIGURE 4. The graphs of $y = x$ and $y = 1 + \frac{1}{2}\tanh(x - 1)$, together with the fixed point iteration starting from $x_0 = 0$.

9. Conclusion

In this paper, we introduced and investigated a Fermatean neutrosophic n-normed linear space. The proposed structure combines the geometric properties of an n-normed linear space with three functions

$$\mu_F, \eta_F, \nu_F : X^n \times (0, \infty) \longrightarrow [0, 1],$$

representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively. These functions were considered under the Fermatean-type admissibility conditions

$$0 \leq \mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 1$$

and

$$0 \leq \mu_F^3(\widetilde{V}, \lambda) + \eta_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 2.$$

Thus, the proposed space retains the cubic flexibility of a Fermatean-type structure while explicitly incorporating an independent indeterminacy function.

Several fundamental concepts of analysis were developed in this framework. In particular, we defined open balls, convergence, Cauchy sequences, and completeness by using the asymptotic behavior of the three Fermatean neutrosophic functions. We proved that the limit of a convergent sequence is unique and established equivalent characterizations of both convergence and Cauchy sequences. We also showed that every convergent sequence is Cauchy.

Concrete examples based on the determinant n-norm on $\mathbb{R}^n$ were provided. These examples demonstrate that the proposed axioms are consistent with the usual geometric interpretation of an n-norm. The truth-membership function tends to one as the associated n-norm tends to zero, whereas the indeterminacy-membership and falsity-membership functions tend to zero.

We further introduced Fermatean neutrosophic n-contraction mappings and established a Banach-type fixed point theorem in complete Fermatean neutrosophic n-normed linear spaces. It was proved that every mapping satisfying the proposed contraction conditions has a unique

fixed point. Moreover, the sequence generated by the Picard iteration

$$x_{k+1} = \tau(x_k)$$

converges to that fixed point.

Finally, the fixed point theorem was applied to affine and nonlinear operator equations. In particular, the affine equation

$$x = c + \alpha x, \qquad |\alpha| < 1,$$

was shown to possess the unique solution

$$x^* = \frac{c}{1-\alpha}.$$

The nonlinear application also demonstrated how a suitable Lipschitz condition leads to the existence and uniqueness of a solution. These applications illustrate the usefulness of the proposed framework in studying operator equations involving uncertainty and indeterminacy.

Consequently, Fermatean neutrosophic n-normed linear spaces provide a flexible extension of fuzzy, intuitionistic fuzzy, Fermatean fuzzy, and neutrosophic normed structures. The theory developed in this paper creates a foundation for further investigations in generalized functional analysis, fixed point theory, approximation theory, and nonlinear operator equations.

10. Suggestions and Further Study

The results obtained in this paper suggest several directions for future research. These directions are supported by related developments in fuzzy, intuitionistic fuzzy, Fermatean fuzzy, and neutrosophic spaces.

(1) **Generalized contraction mappings.**
The fixed point theorem may be extended to Kannan, Chatterjea, Reich, Hardy–Rogers, Geraghty, Meir–Keeler, Ćirić, rational, and integral-type contraction mappings. Generalized fuzzy and intuitionistic fuzzy contractions have already produced important extensions of the Banach contraction principle [33, 34, 38, 39]. Analogous contraction conditions can therefore be investigated in Fermatean neutrosophic n-normed linear spaces.

(2) **Common, coupled, and multidimensional fixed points.**
Common fixed points of two or more mappings may be studied under commutativity, compatibility, weak compatibility, or occasionally weak compatibility assumptions. Coupled fixed point theorems have been established in intuitionistic fuzzy normed spaces [61], while related contraction results have been obtained in complete intuitionistic fuzzy normed linear spaces [39]. These results motivate the investigation of common, coupled, tripled, and multidimensional fixed points in the present framework.

(3) **Best proximity and approximate fixed points.**
When a mapping is not a self-mapping, a fixed point may not exist. Therefore, best proximity points and approximate fixed points may be developed as alternatives to exact fixed points. The contraction techniques used in fuzzy metric spaces [30, 32, 33] may provide a foundation for obtaining such results in Fermatean neutrosophic n-normed linear spaces.

(4) **Alternative t-norms and t-conorms.**
The fixed point theorem proved in this paper is mainly formulated using the minimum t-norm and maximum t-conorm. Future work may consider product, Łukasiewicz, Hamacher, Dombi, Frank, and other continuous t-norms and t-conorms. The importance of these operations in fuzzy and intuitionistic fuzzy spaces can be observed in [35, 36, 21], while Dombi operators in a Fermatean fuzzy environment were investigated in [9].

(5) **General q-power structures.**
The cubic Fermatean-type conditions may be extended to a general q-power structure. For example, one may consider

$$\mu_F^q(\widetilde{V},\lambda)+\nu_F^q(\widetilde{V},\lambda)\le 1, \qquad q\ge 1,$$

together with a suitable admissibility condition involving η_F^q. The progression from intuitionistic fuzzy sets to Pythagorean and Fermatean fuzzy sets [2, 3, 4, 7] suggests that such a generalization could lead to a unified family of q-rung Fermatean neutrosophic n-normed linear spaces.

(6) **Topological properties.**
The topology generated by Fermatean neutrosophic open balls may be investigated in greater detail. Relevant properties include first and second countability, separability, compactness, connectedness, local compactness, total boundedness, and continuity. Earlier investigations of fuzzy and intuitionistic fuzzy topological structures [27, 28, 29, 35, 36] and neutrosophic metric spaces [46] provide useful foundations for this study.

(7) **Completion theory.**
A general construction of the completion of an incomplete Fermatean neutrosophic n-normed linear space should be developed. The completion of intuitionistic fuzzy 2-metric spaces was investigated in [60], while finite-dimensional fuzzy normed spaces were studied in [31]. Similar methods may be adapted to determine whether every Fermatean neutrosophic n-normed linear space can be embedded densely and isometrically into a complete space.

(8) **Bounded and continuous operators.**
Fermatean neutrosophic bounded linear operators, continuous operators, operator norms, dual spaces, and compact operators may be defined and studied. In particular, the relationships among boundedness, continuity, convergence, and completeness should be established. Continuity, boundedness, and Fréchet differentiation for nonlinear operators between neutrosophic normed spaces were studied in [56], providing a useful starting point for such an operator theory.

(9) **Approximation and stability theory.**
The proposed structure may be applied to best approximation, projection methods, iterative approximation, and Hyers–Ulam stability of functional equations. Fixed point methods have already been used to investigate the stability of functional equations in neutrosophic normed spaces [58]. It would therefore be meaningful to formulate corresponding stability results using the Fermatean neutrosophic n-norm.

(10) **Differential and integral equations.**
The fixed point theorem may be applied to ordinary differential equations, fractional differential equations, Volterra integral equations, Fredholm integral equations, and integro-differential equations. Fixed point results in an extended neutrosophic rectangular metric space have been applied to a Caputo fractional differential equation in [62]. This indicates that Fermatean neutrosophic n-normed spaces may also provide an appropriate setting for studying differential and integral equations involving uncertain and indeterminate data.

(11) **Multivalued mappings.**
The proposed theory may be extended to set-valued mappings by defining a Fermatean neutrosophic form of the Hausdorff n-norm. This extension may lead to Nadler-type fixed point theorems, selection results, and applications to differential inclusions. The methods developed for orthogonal neutrosophic contractions in [53] may be useful in formulating suitable multivalued contraction conditions.

(12) **Ordered and orthogonal structures.**

A partial order or an orthogonality relation may be added to a Fermatean neutrosophic n-normed linear space. Fixed point theorems for monotone or orthogonally preserving mappings could then be established. Orthogonal fixed point methods have been successfully studied in neutrosophic metric and 2-metric spaces [53, 57], which provides motivation for this extension.

(13) **Statistical and summability convergence.**
Statistical, ideal, lacunary, Cesàro, Abel, and matrix convergence may be formulated through μ_F, η_F, and ν_F. Statistical and ideal convergence have been studied in intuitionistic fuzzy n-normed spaces [23, 24, 25, 26]. Statistical and matrix convergence in neutrosophic normed spaces have also been investigated in [48, 54]. These results may be extended to the Fermatean neutrosophic n-normed setting.

(14) **Iterative algorithms and convergence rates.**
In addition to the Picard iteration, future studies may examine Mann, Ishikawa, Noor, Krasnosel'skiĭ, and viscosity iteration methods. Their convergence rates, stability properties, and error estimates may be described using the three functions μ_F, η_F, and ν_F. Existing fuzzy contraction principles [34, 39] provide useful tools for analyzing such iterative procedures.

(15) **Practical applications.**
The proposed framework may be applied to optimization, control systems, image processing, pattern recognition, medical diagnosis, risk analysis, and multi-criteria decision-making. Existing Fermatean fuzzy applications include supplier selection [10], medical-material selection [14], electric-vehicle charging-station selection [12], and decision-making under interval-valued Fermatean information [13]. These studies indicate that Fermatean neutrosophic n-normed models may be useful when geometric information, uncertainty, and indeterminacy occur simultaneously.

These research directions indicate that Fermatean neutrosophic n-normed linear spaces have considerable potential for further theoretical and applied development. Future investigations may establish deeper connections with nonlinear functional analysis, operator theory, generalized metric spaces, approximation theory, differential equations, and uncertainty modelling.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986. https://doi.org/10.1016/S0165-0114(86)80034-3.

[3] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *International Journal of Intelligent Systems*, vol. 28, no. 5, pp. 436–452, 2013. https://doi.org/10.1002/int.21584.

[4] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014. https://doi.org/10.1109/TFUZZ.2013.2278989.

[5] T. Senapati and R. R. Yager, "Some new operations over Fermatean fuzzy numbers and application of Fermatean fuzzy WPM in multiple criteria decision making," *Informatica*, vol. 30, no. 2, pp. 391–412, 2019. https://doi.org/10.15388/Informatica.2019.211.

[6] T. Senapati and R. R. Yager, "Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods," *Engineering Applications of Artificial Intelligence*, vol. 85, pp. 112–121, 2019. https://doi.org/10.1016/j.engappai.2019.05.012.

[7] T. Senapati and R. R. Yager, "Fermatean fuzzy sets," *Journal of Ambient Intelligence and Humanized Computing*, vol. 11, pp. 663–674, 2020. https://doi.org/10.1007/s12652-019-01377-0.

[8] D. Liu, Y. Liu, and X. Chen, "Fermatean fuzzy linguistic set and its application in multicriteria decision making," *International Journal of Intelligent Systems*, vol. 34, no. 5, pp. 878–894, 2019. https://doi.org/10.1002/int.22079.

[9] S. B. Aydemir and S. Gündüz, "Fermatean fuzzy TOPSIS method with Dombi aggregation operators and its application in multi-criteria decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 39, pp. 851–869, 2020. https://doi.org/10.3233/JIFS-191763.

[10] M. Keshavarz-Ghorabaee, M. Amiri, M. Hashemi-Tabatabaei, E. K. Zavadskas, Z. Turskis, and J. Antucheviciene, "A new decision-making approach based on Fermatean fuzzy sets and WASPAS for green construction supplier evaluation," *Mathematics*, vol. 8, no. 12, Art. no. 2202, 2020. https://doi.org/10.3390/math8122202.

[11] H. Garg, G. Shahzadi, and M. Akram, "Decision-making analysis based on Fermatean fuzzy Yager aggregation operators with application in COVID-19 testing facility," *Mathematical Problems in Engineering*, vol. 2020, Art. ID 7279027, 2020. https://doi.org/10.1155/2020/7279027.

[12] P. Rani and A. R. Mishra, "Fermatean fuzzy Einstein aggregation operators-based MULTIMOORA method for electric vehicle charging station selection," *Expert Systems with Applications*, vol. 182, Art. no. 115267, 2021. https://doi.org/10.1016/j.eswa.2021.115267.

[13] P. Rani and A. R. Mishra, "Interval-valued Fermatean fuzzy sets with multi-criteria weighted aggregated sum product assessment-based decision analysis framework," *Neural Computing and Applications*, vol. 34, pp. 8051–8067, 2022. https://doi.org/10.1007/s00521-021-06782-1.

[14] M. Kirişci, İ. Demir, and N. Şimşek, "Fermatean fuzzy ELECTRE multi-criteria group decision-making and most suitable biomedical material selection," *Artificial Intelligence in Medicine*, vol. 127, Art. no. 102278, 2022. https://doi.org/10.1016/j.artmed.2022.102278.

[15] G. Büyüközkan, D. Uztürk, and Ö. Ilıcak, "Fermatean fuzzy sets and its extensions: A systematic literature review," *Artificial Intelligence Review*, vol. 57, Art. no. 138, 2024. https://doi.org/10.1007/s10462-024-10761-y.

[16] S. Gähler, "Lineare 2-normierte Räume," *Mathematische Nachrichten*, vol. 28, pp. 1–43, 1964. https://doi.org/10.1002/mana.19640280102.

[17] A. Misiak, "n-inner product spaces," *Mathematische Nachrichten*, vol. 140, pp. 299–319, 1989. https://doi.org/10.1002/mana.19891400121.

[18] H. Gunawan and Mashadi, "On n-normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, pp. 631–639, 2001. https://doi.org/10.1155/S0161171201010675.

[19] R. A. Wibawa-Kusumah and H. Gunawan, "Two equivalent n-norms on the space of p-summable sequences," *Periodica Mathematica Hungarica*, vol. 67, pp. 63–69, 2013. https://doi.org/10.1007/s10998-013-6129-4.

[20] A. L. Narayanan and S. Vijayabalaji, "Fuzzy n-normed linear space," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 24, pp. 3963–3977, 2005. https://doi.org/10.1155/IJMMS.2005.3963.

[21] S. Vijayabalaji, N. Thillaigovindan, and Y. B. Jun, "Intuitionistic fuzzy n-normed linear space," *Bulletin of the Korean Mathematical Society*, vol. 44, no. 2, pp. 291–308, 2007. https://doi.org/10.4134/BKMS.2007.44.2.291.

[22] M. Ahmad and F. Smarandache, "Fermatean Intuitionistic Fuzzy n-Normed Linear Spaces: Convergence, Completeness, and Fixed Point Applications," Neutrosophic Science International Association (NSIA), 2026.

[23] M. Sen and P. Debnath, "Statistical convergence in intuitionistic fuzzy n-normed linear spaces," *Fuzzy Information and Engineering*, vol. 3, pp. 259–273, 2011. https://doi.org/10.1007/s12543-011-0082-9.

[24] P. Debnath, "Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces," *Computers and Mathematics with Applications*, vol. 63, pp. 708–715, 2012. https://doi.org/10.1016/j.camwa.2011.11.034.

[25] N. Konwar and P. Debnath, "I_λ-convergence in intuitionistic fuzzy n-normed linear space," *Annals of Fuzzy Mathematics and Informatics*, vol. 13, no. 1, pp. 91–107, 2017. https://doi.org/10.30948/afmi.2017.13.1.91.

[26] P. Debnath, "Statistical Cesàro summability in intuitionistic fuzzy n-normed linear spaces leading towards Tauberian theorems," *Axioms*, vol. 13, no. 8, Art. no. 557, 2024. https://doi.org/10.3390/axioms13080557.

[27] A. K. Katsaras and D. B. Liu, "Fuzzy vector spaces and fuzzy topological vector spaces," *Journal of Mathematical Analysis and Applications*, vol. 58, pp. 135–146, 1977. https://doi.org/10.1016/0022-247X(77)90233-5.

[28] A. K. Katsaras, "Fuzzy topological vector spaces II," *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 143–154, 1984. https://doi.org/10.1016/0165-0114(84)90034-4.

[29] O. Kaleva and S. Seikkala, "On fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 12, no. 3, pp. 215–229, 1984. https://doi.org/10.1016/0165-0114(84)90069-1.

[30] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, pp. 385–389, 1988. https://doi.org/10.1016/0165-0114(88)90064-4.

[31] C. Felbin, "Finite-dimensional fuzzy normed linear space," *Fuzzy Sets and Systems*, vol. 48, no. 2, pp. 239–248, 1992. https://doi.org/10.1016/0165-0114(92)90338-5.

[32] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994. https://doi.org/10.1016/0165-0114(94)90162-7.

[33] V. Gregori and A. Sapena, "On fixed-point theorems in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 125, pp. 245–252, 2002. https://doi.org/10.1016/S0165-0114(00)00088-9.

[34] D. Miheţ, "A Banach contraction theorem in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 144, pp. 431–439, 2004. https://doi.org/10.1016/S0165-0114(03)00305-1.

[35] J. H. Park, "Intuitionistic fuzzy metric spaces," *Chaos, Solitons and Fractals*, vol. 22, no. 5, pp. 1039–1046, 2004. https://doi.org/10.1016/j.chaos.2004.02.051.

[36] R. Saadati and J. H. Park, "On the intuitionistic fuzzy topological spaces," *Chaos, Solitons and Fractals*, vol. 27, no. 2, pp. 331–344, 2006. https://doi.org/10.1016/j.chaos.2005.03.019.

[37] M. Ahmad and F. Smarandache, "Compactness and Finite-Dimensional Structure of Intuitionistic Algebra Fuzzy Normed Spaces," Neutrosophic Science International Association (NSIA), 2026

[38] C. Alaca, D. Türkoğlu, and C. Yıldız, "Fixed points in intuitionistic fuzzy metric spaces," *Chaos, Solitons and Fractals*, vol. 29, no. 5, pp. 1073–1078, 2006. https://doi.org/10.1016/j.chaos.2005.08.066.

[39] N. Konwar, A. Esi, and P. Debnath, "New fixed point theorems via contraction mappings in complete intuitionistic fuzzy normed linear space," *New Mathematics and Natural Computation*, vol. 15, no. 1, pp. 65–83, 2019. https://doi.org/10.1142/S1793005719500042.

[40] M. Ahmad and F. Smarandache, "Part I—Neutrosophic Graph Theory," in *Advanced Topics in Neutrosophic Mathematics*, p. 7, 2026.

[41] M. Ahmad and F. Smarandache, "Part II—Neutrosophic Algebra," in *Advanced Topics in Neutrosophic Mathematics*, p. 43, 2026.

[42] M. Ahmad and F. Smarandache, "On Neutrosophic Bidirected Graphs and Their Structural Properties," in *Advanced Topics in Neutrosophic Mathematics*, p. 22, 2026.

[43] P. Majumdar and S. K. Samanta, "On similarity and entropy of neutrosophic sets," *Journal of Intelligent and Fuzzy Systems*, vol. 26, no. 3, pp. 1245–1252, 2014. https://doi.org/10.3233/IFS-130810.

[44] J. Ye, "A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets," *Journal of Intelligent and Fuzzy Systems*, vol. 26, no. 5, pp. 2459–2466, 2014. https://doi.org/10.3233/IFS-130916.

[45] P. Liu and Y. Wang, "Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean," *Neural Computing and Applications*, vol. 25, pp. 2001–2010, 2014. https://doi.org/10.1007/s00521-014-1688-8.

[46] M. Kirişci and N. Şimşek, "Neutrosophic metric spaces," *Mathematical Sciences*, vol. 14, pp. 241–248, 2020. https://doi.org/10.1007/s40096-020-00335-8.

[47] M. Ahmad and M. Mursaleen, "Completeness and Cartesian Product in Neutrosophic Rectangular n-Normed Spaces," *Dera Natung Government College Research Journal*, vol. 10, no. 1, pp. 22–40, 2025. https://doi.org/10.56405/dngcrj.2025.10.01.02.

[48] M. Kirişci and N. Şimşek, "Neutrosophic normed spaces and statistical convergence," *The Journal of Analysis*, vol. 28, pp. 1059–1073, 2020. https://doi.org/10.1007/s41478-020-00234-0.

[49] M. Ahmad and M. Mursaleen, "On deferred statistical summability in neutrosophic n-normed linear space," *Filomat*, vol. 39, no. 18, pp. 6123–6148, 2025.https://doi.org/10.2298/FIL2518123A.

[50] M. Ahmad, E. Savaş, and M. Mursaleen, "On deferred I-statistical rough convergence of difference sequences in neutrosophic normed spaces," *Neutrosophic Sets and Systems*, vol. 96, no. 1, p. 14, 2026.https://doi.org/10.5281/zenodo.17102657.

[51] M. Ahmad and F. Smarandache, "Rough I-statistical convergence of order α in neutrosophic normed spaces: Characterizations and applications," in *Advanced Topics in Neutrosophic Mathematics*, p. 91, 2026.https://doi.org/10.5281/zenodo.20562234.

[52] M. Ahmad, M. Mursaleen, and E. Savaş, "Rough I-$\gamma\tau$-statistical convergence of order α in neutrosophic normed spaces," *Journal of Intelligent & Fuzzy Systems*, 2026, Article ID 18758967261437338.https://doi.org/10.1177/18758967261437338.

[53] U. Ishtiaq, K. Javed, F. Uddin, M. de la Sen, K. Ahmed, and M. U. Ali, "Fixed point results in orthogonal neutrosophic metric spaces," *Complexity*, vol. 2021, Art. ID 2809657, 2021. https://doi.org/10.1155/2021/2809657.

[54] V. A. Khan, M. Arshad, and M. D. Khan, "Some results of neutrosophic normed space via Tribonacci convergent sequence spaces," *Journal of Inequalities and Applications*, vol. 2022, Art. no. 42, 2022. https://doi.org/10.1186/s13660-022-02775-3.

[55] M. Ahmad and F. Smarandache, "Tribonacci Ideal and Ideal-Star Convergence in Non-Archimedean Neutrosophic Normed Spaces," Neutrosophic Science International Association (NSIA), 2026.

[56] V. A. Khan and M. D. Khan, "Nonlinear operators between neutrosophic normed spaces and Fréchet differentiation," *Journal of Inequalities and Applications*, vol. 2022, Art. no. 153, 2022. https://doi.org/10.1186/s13660-022-02893-y.

[57] G. Janardhanan et al., "Orthogonal neutrosophic 2-metric spaces," *Journal of Inequalities and Applications*, vol. 2023, 2023. https://doi.org/10.1186/s13660-023-03024-x.

[58] A. Aloqaily, P. Agilan, K. Julietraja, et al., "A novel stability analysis of functional equation in neutrosophic normed spaces," *Boundary Value Problems*, vol. 2024, Art. no. 47, 2024. https://doi.org/10.1186/s13661-024-01854-2.

[59] M. Mursaleen and Q. M. Danish Lohani, "Intuitionistic fuzzy 2-normed space and some related concepts," *Chaos, Solitons and Fractals*, vol. 42, pp. 224–234, 2009. https://doi.org/10.1016/j.chaos.2008.11.006.

[60] M. Mursaleen, Q. M. Danish Lohani, and S. A. Mohiuddine, "Intuitionistic fuzzy 2-metric space and its completion," *Chaos, Solitons and Fractals*, vol. 42, pp. 1258–1265, 2009. https://doi.org/10.1016/j.chaos.2009.03.025.

[61] M. Eshaghi Gordji, H. Baghani, and Y. J. Cho, "Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces," *Mathematical and Computer Modelling*, vol. 54, pp. 1897–1906, 2011. https://doi.org/10.1016/j.mcm.2011.04.014.

[62] G. Mani, M. A. R. M. Antony, Z. D. Mitrović, A. Aloqaily, and N. Mlaiki, "A fixed point result on an extended neutrosophic rectangular metric space with application," *Boundary Value Problems*, vol. 2024, Art. no. 13, 2024. https://doi.org/10.1186/s13661-024-01820-y.

[63] C. Antony Crispin Sweety and R. Jansi, "Fermatean neutrosophic sets," *International Journal of Advanced Research in Computer and Communication Engineering*, vol. 10, no. 6, pp. 24–27, 2021, https://doi.org/10.17148/IJARCCE.2021.10605.

[64] M. Ahmad and F. Smarandache, *Generalized Functional Analysis in Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Spaces: Theory, Convergence, Compactness, and Applications*, Neutrosophic Science International Association (NSIA) Publishing House, Gallup, USA–Guayaquil, Ecuador, 2026. ISBN: 978-1-59973-885-7. Available at: https://fs.unm.edu/NSIA/.

Published in: Mukhtar Ahmad, Florentin Smarandache. **Neutrosophic Functional Analysis. Foundations, Fermatean n-Norms, and Approximation Theory.** Neutrosophic Science International Association (NSIA) Publishing House, Gallup (NM, USA) & Guayaquil (Ecuador), 2026.